# THE ANALYTICAL DETERMINATION OF ELECTRIC RAILWAY SPEED - TIME RELATIONS

By

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B.E. (Union College) 1911
M.S. (University of California) 1917



THESIS

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA

June, 1919

Approved by the
Sub-Committee-in-Charge,

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## INTRODUCTION

The design of a modern electric railway necessarily involves an accurate predetermination of the performance of the completed system when the construction period shall have passed and the traffic begins to move. Consequently, the speed of trains predominates among the factors to be treated. The speed of a train, like that of any moving body, varies as a function of time in accordance with the fundamental laws of mechanics. From these principles, it is possible and practicable to predetermine the speed of a given electrically propelled train at any instant in the course of a run over a known or assumed track. For high speed roads with frequent stops, such as subway systems, the converse problem of determining the time required to attain a particular speed may be of greater importance and of correspondingly more frequent occurrence.



The predeterminations of speed-time relations necessarily depend on the type of motive equipment and therefore must be based upon the characteristics of the motors in the case of electrically propelled trains. The methods, in current use for carrying out these determinations are step-by-step, graphical processes. The most used

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method is that proposed by Mr. C. O. Mailloux in his "Notes on the Plotting of Speed-Time Curves" in the Transactions of the American Institute of Electrical Engineers, Volume XIX (1902), page 984.

The present purpose is to derive formulae, by means of which the speed-time relations may be determined directly by arithmetic substitutions and operations without recourse to graphs other than the characteristic motor curves. Since the continuous current series motor is used almost universally in American electric railway practice, only the formulae applicable to this type of motor are presented although the method of derivation should indicate how analogous formulae, suited to other types of motors, may be derived.

As a foundation for the subsequent derivations, a segregation of the factors involved in electric propulsion of trains is essential.

# I

## FACTORS IN ELECTRIC TRAIN PROPULSION

### A - ENERGY, POWER AND TRACTIVE EFFORT

The movement of a car or train along a track necessitates an expenditure of energy. In the general case, the mechanical energy, supplied to a train, simultaneously serves three well defined purposes, and consequently must be treated as the sum of three distinct components. These components are: first, that dissipated due to friction; second, that stored or liberated as potential energy due to change in the elevation of the train; and third, that stored or liberated as kinetic energy due to change in the speed of the train.

By the application of what is known as dynamic braking, that is, operating the motors as generators while descending grades or while making stops, mechanical energy may be extracted from a moving train, converted to electric energy and returned to the electric distribution system. Physically, the extraction of energy is the reverse of the process of supplying energy. Thus, in calculations, energy extracted must be treated as negative energy supplied. Hence, the energy supplied in a chosen period of time will be positive or negative according to operating conditions.

The dissipation of energy due to friction continues as long as a train is in motion. The dissipated energy can not be recovered as useful mechanical energy in subsequent operations since it passes off as heat or possibly in other, less easily discernible forms. Its name well defines it as lost. The dissipated energy in turn may conveniently

be resolved into three components. These components are: first, that due to friction in traveling along level tangent track, technically referred to as ~~due to~~ train resistance; second, that due to an increment of friction introduced by track curvature, usually spoken of as ~~due to~~ curve resistance; and third, that due to the application of friction brakes, known as ~~due to~~ braking effort.

Potential energy is stored while a train is ascending a grade. During a subsequent descent, potential energy is liberated and either extracted, dissipated or transformed in to kinetic energy. Kinetic energy is stored while the speed of a train is increasing. As the speed later decreases, kinetic energy is liberated and either extracted, dissipated or transformed to potential energy, depending on whether the decrease in speed is due to dynamic braking, friction or to ascent of a grade.

The potential energy of a body such as a train is ~~the same when it is at the same elevation.~~ a function of its elevation only. Also the kinetic energy of a non-moving body ~~at standstill~~ is zero. Therefore all the energy that is supplied to a train during its run between two stations which are at the same elevation, is ultimately dissipated before the train stops unless some of it is recovered by means of dynamic braking.

Although the foregoing discrimination of the forms in which energy is expended is an essential step in the analysis of train propulsion, the term, speed-time relation, accurately implies instantaneous values of speed and time. Consequently the determination of these relations is immediately concerned with ~~instantaneous~~ rates of energy expenditure, consumption and absorption at successive instants. That is to say, it depends upon the input and distribution of mechanical power, for power is the ~~instantaneous~~ rate of energy expenditure, conversion, consumption or

absorption. The mechanical energy input has been divided into three main components. Similarly, the mechanical power input is the sum of the corresponding instantaneous rates of energy dissipation and storage.

The measure of mechanical power input to a moving body is the algebraic product of the applied force and the instantaneous speed at which the point of application of the force is moving. In an electrically propelled train, the mechanical power input is transferred from the motors through gears to the axles at the surface of the latter. Hence the measure of the mechanical power input to an axle is the product of the tangential force at the surface of the axle and the peripheral speed of that surface. However, in railway calculations, it is convenient to use the speed of the train as a basis of reference so far as possible.

The peripheral speed of a point on the tire of a car wheel is the same as the speed of the car provided the wheel does not slip. The peripheral speed of a point on the tire of a wheel is to the speed of a point on the surface of its axle in the ratio of the diameters of the wheel and axle. Therefore the ratio of the train speed to the peripheral speed of the cyclindrical surface of the axle is equal to the ratio of the diameters of wheel and axle.

By the principle of moments, a tangential force at the surface of an axle may be replaced by an equivalent force applied at the surface of a concentric cyclinder of greater radius. The tangential force, which must be applied at a radius equal to the wheel radius, in order to produce the same moment as the actual tangential force applied at the surface

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By the principle of moments, a tangential force at the surface of an axle may be replaced by an equivalent force applied at the surface of a concentric cylinder of greater radius. The tangential force, which must be applied at a radius equal to the wheel radius, in order to produce the same moment as the actual tangential force applied at the surface

of the axle, must be to the latter in the inverse ratio of the diameters of the wheel and axle. This equivalent force, hypothetically acting on the axle at a radius equal to the semi-diameter of a driving wheel, is called the tractive effort.

The above relations can be expressed concisely in algebraic form; thus

$$\left(\begin{array}{l}\text{Mechanical power}\\ \text{input to train}\\ \text{per motor}\end{array}\right) = \left(\begin{array}{l}\text{Tangential force applied}\\ \text{at surface of a driving}\\ \text{axle}\end{array}\right) \text{X} \left(\begin{array}{l}\text{Peripheral speed}\\ \text{of cyclindrical}\\ \text{surface of axle}\end{array}\right)$$

$$= \left(\begin{array}{l}\text{Tangential force applied}\\ \text{at surface of a driving}\\ \text{axle X axle diameter}\\ \div \text{ wheel diameter}\end{array}\right) \text{X} \left(\begin{array}{l}\text{Peripheral speed}\\ \text{of axle surface}\\ \text{X wheel diameter}\\ \div \text{ axle diameter}\end{array}\right)$$

$$= \text{Motor tractive effort X train speed} .$$

The standard American unit of measure of train speed is miles per hour, and that of tractive effort is pounds, avoirdupois. The total tractive effort applied to a train is the sum of the tractive efforts applied at the several driving axles. However, for purposes of comparison and for simplicity in computations, the tractive effort is usually referred to the weight of the train. That is, the tractive effort is spoken of as so many pounds per ton of gross train weight. The phrase, pounds per ton, unfortunately contains a latent ambiguity because the number of pounds in a ton is presumably constant, but there should be no confusion when the expression is confined to railroad parlance.

Because of the proportionality of mechanical power and tractive effort, the applied tractive effort is obviously divisible into several components corresponding to the rates of energy dissipation and storage.

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Because of the proportionality of mechanical power and tractive effort, the applied tractive effort is obviously divisible into several components corresponding to the rates of energy dissipation and storage.

## B - COMPONENTS OF TRACTIVE EFFORT

In general, the tractive effort applied to a train may be resolved into the following components:

1 - Corresponding to the rate of energy dissipation,
  (a) - Train resistance,
  (b) - Curve resistance,
  (c) - Braking effort of friction brakes;

2 - Corresponding to the rate of storage of potential energy,
  (a) - Grade resistance;

3 - Corresponding to the rate of storage of kinetic energy,
  (a) - Accelerating effort.

Train resistance

When a train is moving on level tangent track, there is energy dissipated in rail and journal friction, air resistance, etc. The sum of the forces, corresponding to the rates of energy dissipation due to these causes, is called the train resistance. The magnitude of the train resistance at any instant depends upon the train weight, speed, cross-sectional area, and the number of cars. The relations of these factors have been determined by extensive experiments with different classes of equipment in operations under wide ranges of conditions. From the experimental data, empirical formulae for train resistance have been deduced. One of these, which is quite commonly used, is that developed by Mr. A. H. Armstrong, namely:

$$R = \frac{50}{\sqrt{T}} + 0.03\,V + \frac{0.002\,X}{T}\left(1 + \frac{N-1}{10}\right)V^2 ,$$

in which

$R$ = Train resistance in pounds per ton of gross train weight,
$T$ = Gross weight of train in tons (2000 lbs.),
$V$ = Speed of train in miles per hour,
$X$ = Projected cross-sectional area of train in square feet,
$N$ = Number of cars in train,

$$\frac{50}{\sqrt{T}} \geq 3.5$$

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$T$ = Gross weight of train in tons (2000 lbs.),
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$X$ = Projected cross-sectional area of train in square feet,
$N$ = Number of cars in train,
$\frac{50}{\sqrt{T}} \leq 2.5$

Curve Resistance

When a train is proceeding along a horizontal curve in the track, there is introduced an additional resistance due to the side pressure of the wheel flanges on the rails, the unequal distribution of weight of the train on the two rails, etc. This additional resistance is called curve resistance.

The curve resiscance varies from $C = 0.5\ D$ to $C = 1.5\ D$ pounds per ton of gross train weight, depending upon the condition of the track and wheels and upon the degree of superelevation of the outside rail. D is the degree of the curve; that is, the number of degrees of central angle subtended by a one hundred-foot chord of the circular arc described by the center line of the track. It is clear that, for average conditions, C may be taken as numerically equal to the degree of the curve.

Braking Effort of Friction Brakes

When friction brakes are applied to the wheels of a moving train, energy is dissipated in accordance with the laws of sliding friction of metal on metal under pressure. However, since the pressure of the brake shoes is subject to the immediate control of the operator, the magnitude of the braking effort does not bear any fixed relation to the train speed. In applications of the brakes on long downgrades, the braking effort is kept practically constant; and, when service stops are being made, the braking period is of such short duration that no serious error in computations is introduced by treating the braking effort as constant during this period. Considerations of the safety of the equipment and of the comfort of passengers dictate the

allowable braking effort. In subsequent formulae, the braking effort of friction brakes is represented by B and, like the other components of tractive effort, it is measured in pounds per ton of gross train weight.

Grade Resistance

In mounting a grade, a train absorbs energy which is returnable when the train later descends to a lower elevation. The component of tractive effort, corresponding to the rate of storage, or liberation, of potential energy, is

$$G = 2000 \sin \phi ,$$

where 2000 is the number of pounds in a ton, and $\phi$ is the vertical angle measured upward from the horizontal to the grade line of the track.

Since the per cent grade, 100 tan $\phi$, can readily be obtained from the surveyors' data, it is convenient to express the grade resistance as

$$G = 2000 \sin \left( \tan^{-1} \left( \frac{\text{Per cent grade}}{100} \right) \right) .$$

For light grades, such as are met in steam railroad practice, there is no appreciable error introduced by assuming that

$$\sin \phi = \tan \phi$$

and using the approximation,

$$G \doteq G' = 2000 \left( \frac{\text{Per cent grade}}{100} \right) = 20 \text{ X Per cent grade.}$$

But much heavier grades are common in urban and suburban electric systems and serious error may be introduced in calculations by applying the approximation.

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But much heavier grades are common in urban and suburban electric systems and serious error may be introduced in calculations by applying the approximation.

While train resistance, curve resistance and the application of friction brakes all tend to retard the motion of a train, it is evident that grade resistance will tend to retard or to accelerate depending on whether the train is proceeding uphill or down. This is taken into account in the formulae by assigning to the value of G the positive or negative algebraic sign as dictated by the above considerations. That is to say, the value of G is positive or negative according as that of sin $\phi$ is positive or negative.

Accelerating Effort

If the magnitude of the propelling force applied to a body is greater than that, under the action of which the existing speed of the body will be maintained, the speed of the body will increase; and, conversely, the speed will decrease if the propelling force is insufficient to maintain the extant speed of the body. Consequently, if the tractive effort, applied to the driving wheels of a train, is more than sufficient to overcome the train resistance, curve resistance, braking effort and grade resistance, the balance will operate to accelerate the speed of the train. This component of the tractive effort may be termed the accelerating effort.

The accelerating effort, necessary to produce an acceleration of A miles per hour per second, is usually taken as 100 A pounds per ton of gross train weight. The value of the accelerating effort is positive or negative depending on whether that of A is positive or negative; that is, according as the speed of the train is increasing or decreasing.

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The accelerating effort, necessary to produce an acceleration of $A$ miles per hour per second, is usually taken as $100\ A$ pounds per ton of gross train weight. The value of the accelerating effort is positive or negative depending on whether that of $A$ is positive or negative; that is, according as the speed of the train is increasing or decreasing.

Summary of Components

The algebraic sum of the above five components is the tractive effort applied to the driving axles through the functioning of the motors and gears. That is to say,

$$F = 100\,A + B + C + G + R \;,$$

in which
- $F$ = Tractive effort applied at driving axles,
- $A$ = Acceleration of train speed in miles per hour per second,
- $B$ = Braking effort of friction brakes,
- $C$ = Curve resistance,
- $G$ = Grade resistance,
- $R$ = Train resistance.

The unit of measure of all, except the acceleration A, is pounds (avoir dupois) per ton (2000 lbs.) of gross train weight.

## C - TRAIN ACCELERATION

Transposing F and A in the above tractive effort equation,

$$A = 0.01\,(\,F - B - C - G - R\,)\;.$$

This relation expresses the fact that the acceleration is determined by the values of the applied tractive effort, braking effort, and curve, grade and train resistance.

The acceleration at a given time is the ~~instantaneous time~~ rate of change of the train speed at that time; that is,

$$A = \frac{dV}{dt}\;.$$

Hence
$$\frac{dV}{dt} = 0.01\,(\,F - B - C - G - R\,)\,.$$

Substituting for R its equivalent, given on page 7 above,

$$\frac{dV}{dt} = 0.01\left\{F - B - C - G - \frac{50}{\sqrt{T}} - 0.03\,V - \frac{0.002\,X}{T}\left(1 + \frac{N-1}{10}\right)V^2\right\} \qquad (1)$$

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## 6 - TRAIN ACCELERATION

Transposing F and A in the above tractive effort equation,

$$A = 0.01\ (\ F - B - C - G - R\ )\ .$$

This relation expresses the fact that the acceleration is determined by the values of the applied tractive effort, braking effort, and curve, grade and train resistances.

The acceleration is the instantaneous time rate of change of the train speed; that is,

$$\frac{dV}{dt} = A\ .$$

Hence
$$\frac{dV}{dt} = 0.01\ (\ F - B - C - G - R\ )\ .$$

Substituting for R its equivalent, given on page 7 above,

$$\frac{dV}{dt} = 0.01 \left\{ F - B - C - G - \frac{20}{\sqrt{T}} - 0.03\ V - \frac{0.002\ X}{T} \left( 1 + \frac{N - 1}{10} \right) V^2 \right\} \qquad (11)$$

This relation (1) is the fundamental differential equation of train speed. Its solutions render formulas for the direct predetermination of the speed-time relations for known or assumed service conditions. As has been previously stated, B, C and G are independent of the train speed V. However, the applied tractive effort F may vary as a function of the train speed as will be seen from a consideration of the characteristic curves of railway motors and the usual methods of operating them.

## II

## RAILWAY MOTOR CHARACTERISTICS

The series motor derives its name from the fact that its field circuit is connected in series with the armature circuit as is shown in Fig. 1. It is clear that the motor current, armature current and field current are identical.

The performance of a motor under operating conditions is most conveniently expressed in what are known as characteristic curves. These curves for railway motors are determined from their performance in tests which are usually made at the factory. Fig. 2 shows the characteristic curves of a continuous current series railway motor. For normal voltage applied to the motor terminals, these curves display the relations between the motor current and each of the following factors:

1. Speed of car or train in miles per hour,
2. Tractive effort of motor in pounds,
3. Efficiency of motor and its gears in per cent,
4. Power output of motor with its gears in kilowatts.

It is clear that the relations, expressed by the curves, are all affected more or less by the diameter of the driving wheels and by the reduction ratio of the gears between motor and axle. Also the relation of speed to current is largely dependent upon the motor terminal voltage. For those reasons, the gear ratio, wheel diameter and terminal voltage, for which the curves apply, are stated on the curve sheet.

The speed-current curve shows that the current decreases as the speed increases. This is due to the direct proportion connecting

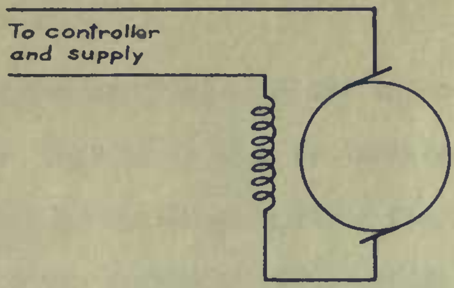


FIG. 1

SCHEMATIC WIRING DIAGRAM
OF
CONTINUOUS CURRENT SERIES MOTOR

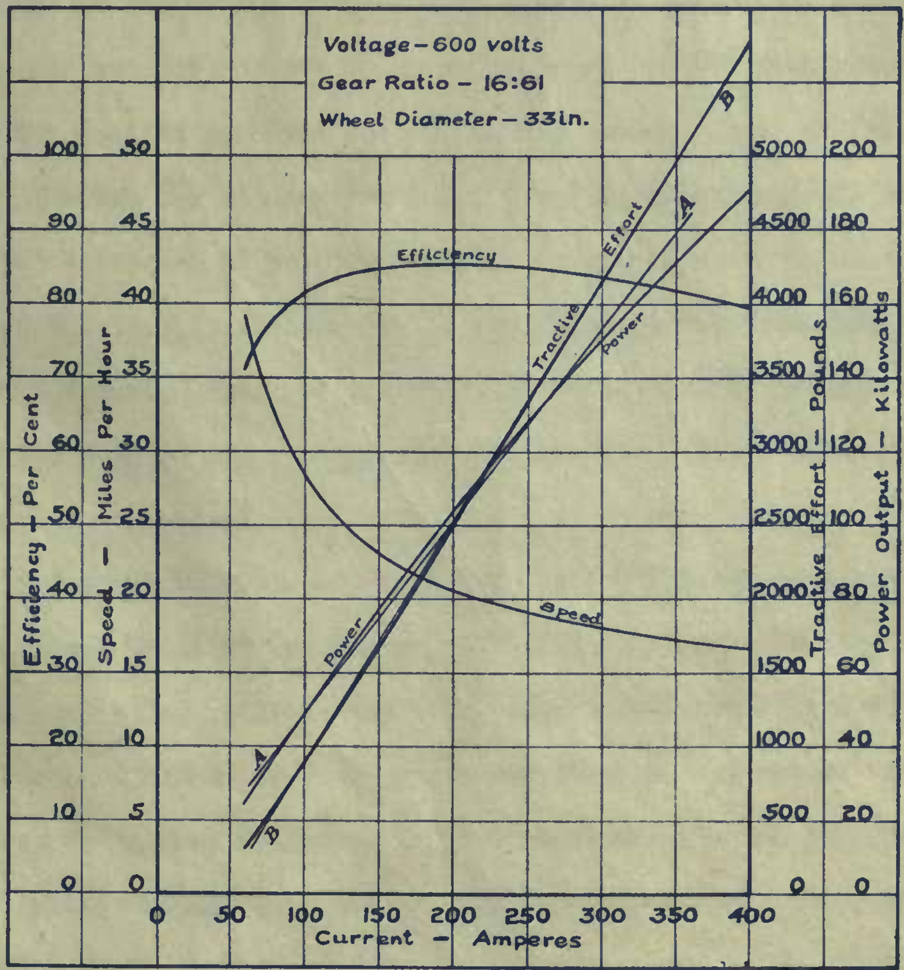


FIG. 2

CHARACTERISTIC CURVES OF A
CONTINUOUS CURRENT SERIES RAILWAY MOTOR

the speed of rotation of the armature with the induced counter-electromotive force of a motor. That is to say, an increase of the armature speed produces an increase of the electromotive force induced in the armature and this opposes the impressed electromotive force, thus reducing the current in the motor circuit. The curves show also that the tractive effort decreases as the current decreases. ~~The~~ as a net result ~~is that~~ the tractive effort input to the driving axles decreases as the speed of the train increases. In other words, the tractive effort bears a fixed relation to the speed as long as constant voltage is applied to the terminals of the motor. This relation must be determined so that an expression for tractive effort in terms of speed can be substituted for F in the differential equation (1) before the latter can be solved.

Although the speed-current and tractive effort-current curves indicate the existence of perfectly definite, continuous relations between speed and tractive effort, it is impossible to express these relations in a formula rationally derived from fundamental principles. The alternative is to use an approximation that is sufficiently exact for engineering purposes.

The tractive effort-current and power output-current curves in Fig. 2 -- and these curves are typical in this respect although they apply to a particular motor -- may be closely approximated by the straight lines AA and BB over the operating range of the motor. It has already been shown that the power input to the train, which is equal to the power outputs of all the motors combined, is the algebraic product of the tractive effort and train speed. By making use of these relations and the above straight line approximations, an algebraic expression of the relation between tractive effort and speed may be obtained.

The equation of the line AA in Fig. 2 is

$$P' = h_1' + h_2'I \tag{2}$$

in which

$P'$ = Power output per motor in kilowatts,
$I$ = Current per motor in amperes,
$h_1'$ and $h_2'$ are constants determined by the co-ordinates of any two points on the line AA.

To reduce the power to kilowatts per ton of gross train weight, substitute

$$\frac{T}{M}P = P' \tag{3}$$

in which

$P$ = mechanical power input to driving axles in kilowatts per ton,
$T$ = Gross weight of train in tons,
$M$ = Number of motors in the train.

Then

$$\frac{T}{M} P = h_1' + h_2'I \tag{4}$$

or

$$P = \frac{M}{T}( h_1' + h_2'I ) \tag{5}$$

Since F is expressed in pounds per ton and V in miles per hour, and since it is desired to express the power in terms of F and V, the power, P kilowatts per ton, must be reduced to mile pounds per hour per ton. One kilowatt is equivalent to

$$503 = \frac{550 \text{ X } 3600}{0.746 \text{ X } 5280} \text{ mile pounds per hour.} \tag{6}$$

Thus the power input to the driving axles is

$$F\,V = 503\,P \text{ mile pounds per hour per ton.} \tag{7}$$

Hence

$$F\,V = \frac{503\,M}{T}( h_1' + h_2'I ) \tag{8}$$

and

$$I = \frac{T\,F\,V - 503\,M\,h_1'}{503\,M\,h_2'} \tag{9}$$

The equation of the line AA in Fig. 2 is

$$P' = h'_1 + h'_2 I \qquad (2)$$

in which

$P'$ = Power output per motor in kilowatts,
$I$ = Current per motor in amperes,
$h'_1$ and $h'_2$ are constants determined by the co-ordinates of any two points on the line AA.

To reduce the power to kilowatts per ton of gross train weight, substitute

$$P' = P \frac{T}{N} \qquad (3)$$

in which

$P$ = mechanical power input to driving axles in kilowatts per ton,
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Then

$$P \frac{T}{N} = h'_1 + h'_2 I \qquad (4)$$

or

$$P = \frac{N}{T} ( h'_1 + h'_2 I ) \qquad (5)$$

Since F is expressed in pounds per ton and V in miles per hour, and since it is desired to express the power in term of F and V, the power, P kilowatts per ton, must be reduced to mile pounds per hour per ton. One kilowatt is equivalent to

$$503 = \frac{550 \times 3600}{0.746 \times 5280} \text{ mile pounds per hour.} \qquad (6)$$

Thus the power input to the driving axles is

$$F V = 503 P \text{ mile pounds per hour per ton.} \qquad (7)$$

Hence

$$F V = \frac{503 N}{T} ( h'_1 + h'_2 I ) \qquad (8)$$

and

$$I = \frac{T F V - 503 N h'_1}{503 N h'_2} \qquad (9)$$

The equation of the straight line BB in Fig. 2 is

$$F' = h_3 + h_4 I \tag{10}$$

in which

$F'$ = Tractive effort output per motor in pounds,
$I$ = Current per motor in amperes,
$h_3$ and $h_4$ are constants determined by the co-ordinates of any two points on the line BB.

To reduce the tractive effort to pounds per ton of gross train weight, substitute

$$\frac{T}{M} F = F' \tag{11}$$

in which

$F$ = Tractive effort input to driving axles in pounds per ton,
$T$ = Gross weight of train in tons,
M $N$ = Number of motors in train.

Then

$$\frac{T}{M} F = h_3 + h_4 I \tag{12}$$

or

$$F = \frac{M}{T}( h_3 + h_4 I ) \ , \tag{13}$$

and

$$I = \frac{T F - M h_3}{M h_4} \ . \tag{14}$$

Combining equations (9) and (14),

$$\frac{T F V - 503 M h_1'}{503 M h_2'} = \frac{T F - M h_3}{M h_4} \ . \tag{15}$$

Then

$$T F V h_4 - 503 M h_1' h_4 = 503 T F h_2' - 503 M h_2' h_3 \ , \tag{16}$$

$$T ( h_4 V - 503 h_2' ) F = 503 M ( h_1' h_4 - h_2' h_3 ) \ , \tag{17}$$

$$F = \frac{503 M ( h_1' h_4 - h_2' h_3)}{T ( h_4 V - 503 h_2')} \ , \tag{18}$$

and

$$F = \frac{503 M}{T} \frac{( h_1' - h_3 h_2' / h_4)}{( V - 503 h_2' / h_4)} \ . \tag{19}$$

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Combining equations (9) and (14),

$$\frac{T F - N h_3}{N h_4} = \frac{T F V - 303 N h_1'}{303 N h_2'} \qquad (15)$$

Then

$$T F V h_4 - 303 N h_1' h_4 = 303 T F h_2' - 303 N h_2' h_3 \qquad (16)$$

$$T \left( h_4 V - 303 h_2' \right) F = 303 N \left( h_1' h_4 - h_2' h_3 \right) \qquad (17)$$

$$F = \frac{303 N \left( h_1' h_4 - h_2' h_3 \right)}{T \left( h_4 V - 303 h_2' \right)} \qquad (18)$$

and

$$F = \frac{303 N}{T} \cdot \frac{\left( h_1' - h_3 h_2' / h_4 \right)}{\left( V - 303 h_2' / h_4 \right)} \qquad (19)$$

Letting $$h_1 = \frac{503\ M}{T}\left( h_1' - h_3\frac{h_2'}{h_4}\right), \qquad (20)$$

and $$h_2 = 503\ \frac{h_2}{h_4} \quad , \qquad (21)$$

$$F = \frac{h_1}{V - h_2} \quad . \qquad (22)$$

This equation (22) is the approximate formula for tractive effort F, in pounds per ton of gross train weight, in terms of the train speed V, in miles per hour, when rated voltage is applied to the terminals of the motors.

For reasons, that will appear presently, the terminal voltage of the motors is reduced below the rated voltage during certain periods in the operation of a train. In fact, there are three conditions of terminal voltage to be considered, namely:

1. Operation at rated voltage;
2. Operation with power shut off, that is, with zero voltage applied;
3. Operation with applied voltage less than rated voltage and not zero.

As has been pointed out, equation (22) applies in the first of these three cases.

While a train is moving with the power shut off, the terminal voltage, applied to the motors, is zero; hence the motor current is zero. The output of the motors, however, is slightly negative. That is, a small amount of energy is extracted from the train and dissipated due to friction in the motors and gears. However the rate of extraction of this energy is so small that, for practical purposes, it is neglected so that

$$F = 0 \qquad (23)$$

Except while starting, trains are seldom operated for any appreciable length of time with any terminal voltage less than normal applied to the motors. In special cases, a constant partial voltage may be applied. But such cases are rare and, when they must be treated, an expression for F in the form of equation (22) can readily be derived by making suitable modifications of the motor characteristic curves to accord with the partial voltage.

Before a train starts, the speed of the motors is zero, so the electromotive force, induced in the armature, is zero. Hence, if voltage is applied to the motor terminals with the train at standstill, the current is limited only by the resistance of the field and armature circuits combined. This resistance is necessarily so small in railway motors that, if the rated voltage were applied to the motor terminals while the train was at standstill, the motors would either be damaged seriously or develop an excessive torque causing the wheels to slip or the train to start with a severe jerk. Therefore the motor terminal voltage is varied so as to keep the current within safe limits during the starting period.

The voltage, supplied to a train, is usually constant so, in controlling continuous current series railway motors, the adjustment of motor terminal voltage is procured by connecting external resistance in series with the motors. From several points of view, the ideal control would maintain the motor current constant at its maximum permissible value throughout the starting period. For practical reasons, however, the custom is to vary the external resistance in a few steps, keeping the current within certain well defined limits while starting. Although

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constant current is not maintained by this method, it is practicable to treat the current as though it were constant at an equivalent average value. It has been shown that the motor tractive effort depends upon the motor current and is independent of the motor terminal voltage except in so far as the latter affects the magnitude of the current. Hence, an equivalent constant starting current produces a corresponding constant tractive effort. In other words, the tractive effort, F in equation (1), is treated as constant from the instant when the train begins to move until the train speed attains the value indicated in the motor characteristic curves as corresponding to the average starting current.

With the permissible average starting current fixed, the corresponding tractive effort is determined from the characteristic curves. This constant tractive effort, expressed in pounds per ton, is the value of F in equation (1) during the starting period.

## III

## SPEED - TIME FORMULAE

### A - TRAIN CYCLE

The trains of a railway system operate in what are called trips between terminals, which may be at the ends of a line of track or intermediate as at the ends of divisions. In the course of a trip, a train usually makes several stops. The period of time, that elapses between when a train leaves one stopping place and when it leaves the next, is called a train cycle.

The first event in a train cycle is the start, which is followed by successive periods of operation each involving more or less different conditions, and finally comes the stop. All the normal conditions of operation, that are met in a train cycle, may be classified under a few typical phases which are named largely in accordance with the effects that they produce upon the train speed. They are:

1. Starting,
2. Accelerating,
3. Running at constant speed,
4. Coasting,
5. Braking,
6. Stop.

Each of these phases of operation entails a unique interpretation of certain terms, particularly the input tractive effort F, in the fundamental differential equation (1). And upon these interpretations rest the solutions of this equation and thereby the formulae for the speed-time relations.

For obvious reasons, the brakes are not ordinarily applied while the motors are functioning and consequently, during the greater part of any train cycle,

$$B = 0 .$$

Also curves and grades are of incidental occurrence so most train cycles include periods when

$$C = 0, \quad G = 0 \quad \text{or} \quad C = G = 0 .$$

When B, C or G is not zero, it may be constant; but, if it is not constant, it is usually necessary and sufficiently accurate for practical purposes to treat it as constant by taking average values over definite intervals. The terms, B, C and G, will be carried as constants in the present derivations in order that the resulting formulae may be more generally universally applicable.

Having determined the tractive effort for the three possible conditions of applied motor voltage, and having divided the train cycle into its component phases of operation, it remains to solve the above equation (1) for each of these several phases.

## B - STARTING

It has been pointed out above that, during the starting period, the voltage applied to the motor terminals is varied so as to maintain practically constant tractive effort. This permits the solution of equation (1) on the assumption that F is constant.

In order to facilitate manipulation, group the constants and, for the starting period, let

$$\alpha_1 = 0.01\left(F - B - C - G - \tfrac{50}{\sqrt{T}}\right) , \qquad (24)$$

$$\beta_1 = 0.0003 , \qquad (25)$$

$$\gamma_1 = \frac{0.00002\,X}{T}\left(1 + \frac{N-1}{10}\right) . \qquad (26)$$

Then equation (1) becomes

$$\frac{dV}{dt} = \alpha_1 - \beta_1 V - \gamma_1 V^2 , \qquad (27)$$

$$dt = \frac{dV}{\alpha_1 - \beta_1 V - \gamma_1 V^2} , \qquad (28)$$

$$\gamma_1 dt = \frac{dV}{\frac{\alpha_1}{\gamma_1} - \frac{\beta_1}{\gamma_1}V - V^2} \qquad (29)$$

$$= \frac{dV}{\frac{\alpha_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2} - \frac{\beta_1^2}{4\gamma_1^2} - \frac{\beta_1}{\gamma_1}V - V^2} \qquad (30)$$

$$= \frac{dV}{\left(\frac{\alpha_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}\right) - \left(\frac{\beta_1}{2\gamma_1} + V\right)^2} . \qquad (31)$$

Since $\frac{\beta_1}{2\gamma_1}$ is constant,

$$d\left(\frac{\beta_1}{2\gamma_1} + V\right) = dV \qquad (32)$$

so that

$$\gamma_1 dt = \frac{d\left(\frac{\beta_1}{2\gamma_1} + V\right)}{\left(\frac{\alpha_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}\right) - \left(\frac{\beta_1}{2\gamma_1} + V\right)^2} \qquad (33)$$

Integration of this equation renders

$$\gamma_1 t = \frac{1}{2\sqrt{\frac{\alpha_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}}} \log_e \left[\frac{\sqrt{\frac{\alpha_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}} + \left(\frac{\beta_1}{2\gamma_1} + V\right)}{\sqrt{\frac{\alpha_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}} - \left(\frac{\beta_1}{2\gamma_1} + V\right)}\right] + C_1' \qquad (34)$$

where $C_1'$ is a constant of integration. Dividing both members of equation (34) by $\gamma_1$,

$$t = \frac{1}{\sqrt{4\alpha_1\gamma_1 + \beta_1^2}} \log_e \left[\frac{\sqrt{4\alpha_1\gamma_1 + \beta_1^2} + \beta_1 + 2\gamma_1 V}{\sqrt{4\alpha_1\gamma_1 + \beta_1^2} - \beta_1 - 2\gamma_1 V}\right] + \frac{C_1'}{\gamma_1} \qquad (35)$$

Transforming to common logarithms in order to facilitate computation,

$$t = \frac{2.30}{\sqrt{4\alpha_1\gamma_1 + \beta_1^2}} \log_{10} \left[\frac{\sqrt{4\alpha_1\gamma_1 + \beta_1^2} + \beta_1 + 2\gamma_1 V}{\sqrt{4\alpha_1\gamma_1 + \beta_1^2} - \beta_1 - 2\gamma_1 V}\right] + \frac{C_1'}{\gamma_1} \qquad (36)$$

In order to facilitate manipulation, group the constants and, for the starting period, let

$$a_1 = 0.01\left(F - B - C - G - \frac{SQ}{T}\right) , \quad (24)$$

$$\beta_1 = 0.0003 , \quad (25)$$

$$\gamma_1 = \frac{0.00002\,X}{T}\left(1 + \frac{N-1}{10}\right) . \quad (26)$$

Then equation (1) becomes

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Integration of this equation renders

$$\gamma_1 t = \frac{1}{2\sqrt{\frac{a_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}}} \log_e \left[\frac{\sqrt{\frac{a_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}} + \left(\frac{\beta_1}{2\gamma_1} + V\right)}{\sqrt{\frac{a_1}{\gamma_1} + \frac{\beta_1^2}{4\gamma_1^2}} - \left(\frac{\beta_1}{2\gamma_1} + V\right)}\right] + C_1' \quad (34)$$

where $C_1'$ is a constant of integration. Dividing both members of equation (34) by $\gamma_1$,

$$t = \frac{1}{\sqrt{4a_1\gamma_1 + \beta_1^2}} \log_e \left[\frac{\sqrt{4a_1\gamma_1 + \beta_1^2} + \beta_1 + 2\gamma_1 V}{\sqrt{4a_1\gamma_1 + \beta_1^2} - \beta_1 - 2\gamma_1 V}\right] + \frac{C_1'}{\gamma_1} \quad (35)$$

Transforming to common logarithms in order to facilitate computation,

$$t = \frac{2.30}{\sqrt{4a_1\gamma_1 + \beta_1^2}} \log_{10} \left[\frac{\sqrt{4a_1\gamma_1 + \beta_1^2} + \beta_1 + 2\gamma_1 V}{\sqrt{4a_1\gamma_1 + \beta_1^2} - \beta_1 - 2\gamma_1 V}\right] + \frac{C_1'}{\gamma_1} \quad (36)$$

If the train speed is $V_0$ at the instant $t_0$; the time $t$, in the starting period, at which the speed will reach any other value $V$, is given by

$$t - t_0 = \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}} \log_{10}\left[\frac{\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V}\right] - \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}} \log_{10}\left[\frac{\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V_0}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V_0}\right]; \qquad (37)$$

that is,

$$t = t_0 + \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\left\{\log_{10}\left[\frac{\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V}\right] - \log_{10}\left[\frac{\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V_0}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V_0}\right]\right\} \qquad (38)$$

or

$$t = t_0 + \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\left[\begin{array}{l}\log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V\right) - \log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V\right) \\ \quad - \log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V_0\right) + \log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V_0\right)\end{array}\right] \qquad (39)$$

where $t$ and $t_0$ are expressed in seconds and $V$ and $V_0$ in miles per hour.

Combining equations (36) and (38), and letting

$$C_1 = \frac{C_1'}{\gamma_1} \qquad (40)$$

it is seen that

$$C_1 = \frac{C_1'}{\gamma_1} = t_0 - \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}} \log_{10}\left[\frac{\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V_0}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V_0}\right] \qquad (41)$$

or

$$C_1 = t_0 + \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\left[\log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V_0\right) - \log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V_0\right)\right] \qquad (42)$$

and equation (39) reduces to

$$t = C_1 + \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\left[\log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V\right) - \log_{10}\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V\right)\right]. \qquad (43)$$

<u>Special Case; $V_0 = 0$ , $t_0 = 0$</u>

If a train is started from standstill and time is measured from the instant of starting, so that $V_0 = 0$ and $t_0 = 0$ ; equation (38) shows that the number of seconds, required for the train to attain the speed $V$ miles per hour, is

$$t_s = \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\log_{10}\left[\frac{\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V}\right] - \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\log_{10}\left[\frac{\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1}\right] \qquad (44)$$

$$= \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\log_{10}\left[\frac{\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1+2\gamma_1 V\right)\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1\right)}{\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}-\beta_1-2\gamma_1 V\right)\left(\sqrt{4\alpha_1\gamma_1+\beta_1^2}+\beta_1\right)}\right] \qquad (45)$$

$$= \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\log_{10}\left[\frac{2\alpha_1-\beta_1 V+V\sqrt{4\alpha_1\gamma_1+\beta_1^2}}{2\alpha_1-\beta_1 V-V\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\right] \qquad (46)$$

$$t_s = \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\left[\log_{10}\left(2\alpha_1-\beta_1 V+V\sqrt{4\alpha_1\gamma_1+\beta_1^2}\right) - \log_{10}\left(2\alpha_1-\beta_1 V-V\sqrt{4\alpha_1\gamma_1+\beta_1^2}\right)\right] \qquad (47)$$

## C - NORMAL TERMINAL VOLTAGE APPLIED TO MOTORS

At the conclusion of the above starting period, when full rated voltage is applied to the motor terminals, the motor proceeds to function in accordance with its characteristic curves. Ordinarily, there will be a period of acceleration between the end of the starting period and the time when full speed is attained. Also, in subsequent periods of a train cycle, there may be acceleration, positive or negative, as grades, curves, etc. are encountered. As long as rated terminal voltage is applied to the motors, the relation of tractive effort and speed is approximately

$$F = \frac{h_1}{V-h_2} \quad . \qquad (22)$$

Substituting this in equation (1) gives

$$\frac{dV}{dt} = 0.01\left[\frac{h_1}{V-h_2} - B - C - G - \frac{50}{\sqrt{T}} - 0.03V - \frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right] \qquad (48)$$

$$= \frac{0.01}{V-h_2}\left\{\begin{array}{l} h_1 + h_2\left(B+C+G+\frac{50}{\sqrt{T}}\right) \\ +\left[0.03h_2 - \left(B+C+G+\frac{50}{\sqrt{T}}\right)\right]V \\ +\left[\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)h_2 - 0.03\right]V^2 \\ +\left[\frac{-0.002X}{T}\left(1+\frac{N-1}{10}\right)\right]V^3 \end{array}\right\} \qquad (49)$$

$$t_s = \frac{2.30}{\sqrt{4a_1T+B_1^2}}\log_{10}\left[\frac{\sqrt{4a_1T+B_1^2}+B_1+2TV}{\sqrt{4a_1T+B_1^2}-B_1-2TV}\right] - \frac{2.30}{\sqrt{4a_1T+B_1^2}}\log_{10}\left[\frac{\sqrt{4a_1T+B_1^2}+B_1}{\sqrt{4a_1T+B_1^2}-B_1}\right] \quad (44)$$

$$= \frac{2.30}{\sqrt{4a_1T+B_1^2}}\log_{10}\left[\frac{\left(\sqrt{4a_1T+B_1^2}+B_1+2TV\right)\left(\sqrt{4a_1T+B_1^2}-B_1\right)}{\left(\sqrt{4a_1T+B_1^2}-B_1-2TV\right)\left(\sqrt{4a_1T+B_1^2}+B_1\right)}\right] \quad (45)$$

$$= \frac{2.30}{\sqrt{4a_1T+B_1^2}}\log_{10}\left[\frac{2a_1-B_1V+V\sqrt{4a_1T+B_1^2}}{2a_1-B_1V-V\sqrt{4a_1T+B_1^2}}\right] \quad (46)$$

$$t_s = \frac{2.30}{\sqrt{4a_1T+B_1^2}}\left[\log_{10}\left(2a_1-B_1V+V\sqrt{4a_1T+B_1^2}\right) - \log_{10}\left(2a_1-B_1V-V\sqrt{4a_1T+B_1^2}\right)\right] \quad (47)$$

## D - NORMAL TERMINAL VOLTAGE APPLIED TO MOTORS

At the conclusion of the above starting period, when full rated voltage is applied to the motor terminals, the motor proceeds to function in accordance with its characteristic curves. Ordinarily, there will be a period of acceleration between the end of the starting period and the time when full speed is attained. Also, in subsequent periods of a train cycle, there may be acceleration, positive or negative, as grades, curves, etc. are encountered. As long as rated terminal voltage is applied to the motors, the relation of tractive effort and speed is approximately

$$F = \frac{h_1}{V-h_2} \quad (22)$$

Substituting this in equation (1) gives

$$\frac{dV}{dt} = 0.01\left[\frac{h_1}{V-h_2} - B - C - G - \frac{50}{W} - 0.03V - \frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right] \quad (48)$$

$$= \frac{0.01}{V-h_2}\left\{\begin{array}{l} h_1 + h_2\left(B+C+G+\frac{50}{W}\right) \\ +\left[0.03h_2 - \left(B+C+G+\frac{50}{W}\right)\right]V \\ +\left[\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)h_2 - 0.03\right]V^2 \\ +\left[-\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)\right]V^3 \end{array}\right\} \quad (49)$$

Let
$$\delta_2 = 0.01\left[\frac{-0.002\,X}{T}\left(1+\frac{N-1}{10}\right)\right] . \tag{50}$$

Then
$$\frac{dV}{dt} = \frac{\delta_2}{V-h_2}\left\{\begin{aligned}&\frac{0.01}{\delta_2}\left[h_1 + h_2\left(B+C+G+\frac{50}{\sqrt{T}}\right)\right] + \frac{0.01}{\delta_2}\left[0.03h_2 - \left(B+C+G+\frac{50}{\sqrt{T}}\right)\right]V \\ &+ \frac{0.01}{\delta_2}\left[\frac{0.002\,X}{T}\left(1+\frac{N-1}{10}\right)h_2 - 0.03\right]V^2 + V^3\end{aligned}\right\} . \tag{51}$$

Let
$$\alpha_2 = \frac{0.01}{\delta_2}\left[h_1 + h_2\left(B+C+G+\frac{50}{\sqrt{T}}\right)\right] , \tag{52}$$

$$\beta_2 = \frac{0.01}{\delta_2}\left[0.03h_2 - \left(B+C+G+\frac{50}{\sqrt{T}}\right)\right], \tag{53}$$

$$\gamma_2 = \frac{0.01}{\delta_2}\left[\frac{0.002\,X}{T}\left(1+\frac{N-1}{10}\right)h_2 - 0.03\right] . \tag{54}$$

Then
$$\frac{dV}{dt} = \frac{\delta_2}{V-h_2}\left(\alpha_2 + \beta_2 V + \gamma_2 V^2 + V^3\right) . \tag{55}$$

## 1 - Balancing Speed

In railway parlance, a train operating at constant speed with rated voltage applied to the motor terminals is spoken of as running at balancing speed. The term arises from the fact that when a train ~~is~~ in operating at constant speed, the power input is just balanced by the power dissipated plus the rate of storage of potential energy. In other words, there is no kinetic energy being stored in, or extracted from, the train.

Since the speed, V in the foregoing equations, becomes constant at the balancing speed, the latter can be designated as a particular value of V, say $V_2$ .

Also, when the speed is constant, the rate of change of speed is zero; that is, when $V = V_2$ , $\frac{dV}{dt} = 0$ . (56)

Then, from equation (55),

$$\frac{\delta_2}{V_2 - h_2}(\alpha_2 + \beta_2 V_2 + \gamma_2 V_2^2 + V_2^3) = 0 \ . \tag{57}$$

Dividing this by $\frac{\delta_2}{V_2 - h_2}$,

$$V_2^3 + \gamma_2 V_2^2 + \beta_2 V_2 + \alpha_2 = 0 \ . \tag{58}$$

In order to determine the value of $V_2$,

let
$$V_2 = W - \frac{\gamma_2}{3} \ . \tag{59}$$

Then
$$W^3 + (\beta_2 - \frac{\gamma_2^2}{3})W + (\alpha_2 - \frac{\beta_2\gamma_2}{3} + \frac{2\gamma_2^3}{27}) = 0 \ . \tag{60}$$

Let
$$q = \beta_2 - \frac{\gamma_2^2}{3} \tag{61}$$

and
$$r = \alpha_2 - \frac{\beta_2\gamma_2}{3} + \frac{2\gamma_2^3}{27} \ . \tag{62}$$

Then
$$W^3 + qW + r = 0 \ . \tag{63}$$

Let
$$W = y + z \quad \text{where} \quad y = \frac{-q}{3z} \ ; \tag{64}$$

that is, let
$$W = z - \frac{q}{3z} \ . \tag{65}$$

Then
$$(z - \frac{q}{3z})^3 + q(z - \frac{q}{3z}) + r = 0 \tag{66}$$

or
$$z^3 - qz + \frac{q^2}{3z} - \frac{q^3}{27z^3} + qz - \frac{q^2}{3z} + r = 0 \ ; \tag{67}$$

that is,
$$z^3 + r - \frac{q^3}{27z^3} = 0 \ , \tag{68}$$

and
$$z^6 + rz^3 - \frac{q^3}{27} = 0 \ , \tag{69}$$

whence
$$z^3 = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \ . \tag{70}$$

Now
$$y^3 = \frac{-q^3}{27z^3} = -\frac{r}{2} \mp \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \ , \tag{71}$$

so that
$$y + z = \left[-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right]^{1/3} + \left[-\frac{r}{2} \mp \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right]^{1/3} \tag{72}$$

or, by (64),
$$W = \left[-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right]^{1/3} + \left[-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right]^{1/3} \ . \tag{73}$$

Then, from equation (55),

$$\frac{\delta_2}{V_2-h_2}\left(\alpha_2+\beta_2 V_2+\gamma_2 V_2^2+V_2^3\right)=0 . \tag{57}$$

Dividing this by $\frac{\delta_2}{V_2-h_2}$,

$$V_2^3+\gamma_2 V_2^2+\beta_2 V_2+\alpha_2 = 0 . \tag{58}$$

In order to determine the value of $V_2$:

Let
$$V_2 = W-\frac{\gamma_2}{3} . \tag{59}$$

Then
$$W^3+\left(\beta_2-\frac{\gamma_2^2}{3}\right)W+\left(\alpha_2-\frac{\beta_2\gamma_2}{3}+\frac{2\gamma_2^3}{27}\right)=0 . \tag{60}$$

Let
$$q=\beta_2-\frac{\gamma_2^2}{3} \tag{61}$$

and
$$r=\alpha_2-\frac{\beta_2\gamma_2}{3}+\frac{2\gamma_2^3}{27} . \tag{62}$$

Then
$$W^3+qW+r=0 . \tag{63}$$

Let
$$W=y+z \quad \text{where} \quad y=\frac{-q}{3z} ; \tag{64}$$

that is, let
$$W=z-\frac{q}{3z} . \tag{65}$$

Then
$$\left(z-\frac{q}{3z}\right)^3+q\left(z-\frac{q}{3z}\right)+r=0 \tag{66}$$

or
$$z^3-qz+\frac{q^2}{3z}-\frac{q^3}{27z^3}+qz-\frac{q^2}{3z}+r=0 ; \tag{67}$$

that is,
$$z^3+r-\frac{q^3}{27z^3}=0 , \tag{68}$$

and
$$z^6+rz^3-\frac{q^3}{27}=0 , \tag{69}$$

whence
$$z^3=-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}} . \tag{70}$$

Now
$$y^3=\frac{-q^3}{27z^3}=-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}+\frac{q^3}{27}} , \tag{71}$$

so that
$$y+z=\left[-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right]^{1/3}+\left[-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right]^{1/3} \tag{72}$$

or, by (64),
$$W=\left[-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right]^{1/3}+\left[-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right]^{1/3} . \tag{73}$$

In general, W will have three distinct values, at least one of which is real. That is,

$$W^3 + qW + r = 0 \tag{63}$$

has three roots, and at least one root is real.

Let m be a real value of

$$\left[-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right]^{1/3} , \tag{74}$$

and let n be a real value of

$$\left[-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right]^{1/3} . \tag{75}$$

Then m + n will be a real root of W.

Let $\omega = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$ , $\omega^2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$ , $j = +\sqrt{-1}$ . (76)

Then the roots of

$$W^3 + qW + r = 0 \tag{63}$$

are

$$m+n , \quad \omega m + \omega^2 n , \quad \omega^2 m + \omega n . \tag{77}$$

And, since

$$V_2 = W - \frac{\gamma_2}{3} , \tag{59}$$

the values of $V_2$ in

$$V_2^3 + \gamma_2 V_2^2 + \beta_2 V_2 + \alpha_2 = 0 \tag{58}$$

are

$$\rho_1 = -\frac{\gamma_2}{3} + m + n , \tag{78}$$

$$\rho_2 = -\frac{\gamma_2}{3} + \omega m + \omega^2 n , \tag{79}$$

$$\rho_3 = -\frac{\gamma_2}{3} + \omega^2 m + \omega n . \tag{80}$$

It is seen, from equation (73) and the roots (77), that,

if

$$\frac{r^2}{4} + \frac{q^3}{27} = 0 , \quad < 0 \text{ roots real also} \tag{81}$$

$$m = n \tag{82}$$

and the values of W are $2m, -m, -m,$ all real. (83)

That is, all the values of W, and consequently of $V_2$ would be real

which might lead to ambiguity. However, in the problem at hand, an investigation of the ranges of values of

$$q = \beta_2 - \frac{\gamma_2^2}{3} \tag{61}$$

and

$$r = \alpha_2 - \frac{\beta_2\gamma_2}{3} + \frac{\gamma_2^3}{27} \tag{62}$$

shows that, in ordinary cases,

$$\frac{r^2}{4} + \frac{q^3}{27} \neq 0 \ . \tag{84}$$

In other words, the solution yields only one real value of $V_2$ and two imaginary values. The real, or principal, value of $V_2$ is the actual balancing speed.

## 2 - Acceleration

The next step is to derive a formula by which the speed-time relations can be determined for periods when the speed of the train is increasing or decreasing with rated terminal voltage applied to the motors.

In the preceding section, the fundamental equation (1) was adapted to the condition for full voltage applied to the motor terminals, and reduced to simplest terms, namely:

$$\frac{dV}{dt} = \frac{\delta_2}{V - h_2}(\alpha_2 + \beta_2 V + \gamma_2 V^2 + V^3) \ . \tag{55}$$

Then it was shown that, when

either

$$V_2 = \rho_1 = -\frac{\gamma_2}{3} + m + n \ , \tag{78}$$

$$V_2 = \rho_2 = -\frac{\gamma_2}{3} + \omega m + \omega^2 n \ , \tag{79}$$

or

$$V_2 = \rho_3 = -\frac{\gamma_2}{3} + \omega^2 m + \omega n \tag{80}$$

were substituted for V in

$$V^3 + \gamma_2 V^2 + \beta_2 V + \alpha_2 \ ,$$

then

$$V_2^3 + \gamma_2 V_2^2 + \beta_2 V_2 + \alpha_2 = 0 \ . \tag{58}$$

which might lead to ambiguity. However, in the problem at hand, an investigation of the ranges of values of

$$q = \beta_2 - \frac{\gamma_2^2}{3} \tag{61}$$

and

$$r = \alpha_2 - \frac{\beta_2\gamma_2}{3} + \frac{2\gamma_2^3}{27} \tag{62}$$

shows that, in ordinary cases,

$$\frac{r^2}{4} + \frac{q^3}{27} \neq 0 \; . \tag{48}$$

In other words, the solution yields only one real value of $V_2$ and two imaginary values. The real, or principal, value of $V_2$ is the actual balancing speed.

## <u>2 - Acceleration</u>

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$$\frac{dV}{dt} = \frac{\delta_2}{V-h_2}(\alpha_2+\beta_2V+\gamma_2V^2+V^3) \; . \tag{55}$$

Then it was shown that, when

either $$V_2 = \rho_1 = -\frac{\gamma_2}{3}+m+n \; , \tag{78}$$

$$V_2 = \rho_2 = -\frac{\gamma_2}{3}+\omega m+\omega^2 n \; , \tag{79}$$

or $$V_2 = \rho_3 = -\frac{\gamma_2}{3}+\omega^2 m+\omega n \tag{80}$$

were substituted for V in

$$V^3+\gamma_2V^2+\beta_2V+\alpha_2 \; ,$$

then $$V_2^3+\gamma_2V_2^2+\beta_2V_2+\alpha_2 = 0 \; . \tag{58}$$

Therefore

$$V^3 + \gamma_2 V^2 + \beta_2 V + \alpha_2 = (V-\rho_1)(V-\rho_2)(V-\rho_3) . \tag{85}$$

Also

$$\alpha_2 = -\rho_1\rho_2\rho_3 \ , \tag{86}$$

$$\beta_2 = \rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 \ , \tag{87}$$

$$\gamma_2 = -(\rho_1 + \rho_2 + \rho_3) \ . \tag{88}$$

Introducing the relation (85) into equation (55) renders

$$\frac{dV}{dt} = \frac{\delta_2}{V-h_2}(V-\rho_1)(V-\rho_2)(V-\rho_3) \tag{89}$$

which transposed is

$$dt.\delta_2 = \frac{(V-h_2)dV}{(V-\rho_1)(V-\rho_2)(V-\rho_3)} . \tag{90}$$

Let

$$f(V) = V - h_2 \tag{91}$$

and

$$g(V) = (V-\rho_1)(V-\rho_2)(V-\rho_3) = V^3 + \gamma_2 V^2 + \beta_2 V + \alpha_2 \ , \tag{92}$$

whence

$$g'(V) = \frac{dg}{dV} = 3V^2 + 2\gamma_2 V + \beta_2 \ . \tag{93}$$

That is,

$$\frac{f(V)}{g(V)} = \frac{V-h_2}{(V-\rho_1)(V-\rho_2)(V-\rho_3)} \ , \tag{94}$$

which may be resolved into partial fractions, thus

$$\frac{f(V)}{g(V)} = \frac{l_2}{V-\rho_1} + \frac{m_2}{V-\rho_2} + \frac{n_2}{V-\rho_3} \ , \tag{95}$$

where

$$l_2 = \frac{f(\rho_1)}{g'(\rho_1)} = \frac{\rho_1 - h_2}{3\rho_1^2 + 2\gamma_2\rho_1 + \beta_2} \ , \tag{96}$$

$$m_2 = \frac{f(\rho_2)}{g'(\rho_2)} = \frac{\rho_2 - h_2}{3\rho_2^2 + 2\gamma_2\rho_2 + \beta_2} \ , \tag{97}$$

$$n_2 = \frac{f(\rho_3)}{g'(\rho_3)} = \frac{\rho_3 - h_2}{3\rho_3^2 + 2\gamma_2\rho_3 + \beta_2} \ . \tag{98}$$

From the equations (90), (94) and (95), it is seen that

$$dt.\delta_2 = \frac{l_2\,dV}{V-\rho_1} + \frac{m_2\,dV}{V-\rho_2} + \frac{n_2\,dV}{V-\rho_3} \ . \tag{99}$$

Integration then renders

$$t.\delta_2 = l_2 \log_e(V-\rho_1) + m_2 \log_e(V-\rho_2) + n_2 \log_e(V-\rho_3) + C_2' \tag{100}$$

where $C_2'$ is the constant of integration.

Therefore

$$V^3 + \gamma_2 V^2 + \beta_2 V + \alpha_2 = (V-\rho_1)(V-\rho_2)(V-\rho_3) . \quad (85)$$

Also

$$\alpha_2 = -\rho_1\rho_2\rho_3 , \quad (86)$$

$$\beta_2 = \rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 , \quad (87)$$

$$\gamma_2 = -(\rho_1+\rho_2+\rho_3) . \quad (88)$$

Introducing the relation (85) into equation (83) renders

$$\frac{dV}{dt} = \frac{\delta_2}{V-h_2}(V-\rho_1)(V-\rho_2)(V-\rho_3) \quad (89)$$

which transposed is

$$dt.\delta_2 = \frac{(V-h_2)dV}{(V-\rho_1)(V-\rho_2)(V-\rho_3)} . \quad (90)$$

Let

$$f(V) = V-h_2 \quad (91)$$

and

$$g(V) = (V-\rho_1)(V-\rho_2)(V-\rho_3) = V^3+\gamma_2V^2+\beta_2V+\alpha_2 , \quad (92)$$

whence

$$g'(V) = \frac{dg}{dV} = 3V^2+2\gamma_2V+\beta_2 . \quad (93)$$

That is,

$$\frac{f(V)}{g(V)} = \frac{V-h_2}{(V-\rho_1)(V-\rho_2)(V-\rho_3)} , \quad (94)$$

which may be resolved into partial fractions, thus

$$\frac{f(V)}{g(V)} = \frac{l_2}{V-\rho_1} + \frac{m_2}{V-\rho_2} + \frac{n_2}{V-\rho_3} , \quad (95)$$

where

$$l_2 = \frac{f(\rho_1)}{g'(\rho_1)} = \frac{\rho_1-h_2}{3\rho_1^2+2\gamma_2\rho_1+\beta_2} , \quad (96)$$

$$m_2 = \frac{f(\rho_2)}{g'(\rho_2)} = \frac{\rho_2-h_2}{3\rho_2^2+2\gamma_2\rho_2+\beta_2} , \quad (97)$$

$$n_2 = \frac{f(\rho_3)}{g'(\rho_3)} = \frac{\rho_3-h_2}{3\rho_3^2+2\gamma_2\rho_3+\beta_2} . \quad (98)$$

From the equations (90), (94) and (95), it is seen that

$$dt.\delta_2 = \frac{l_2dV}{V-\rho_1} + \frac{m_2dV}{V-\rho_2} + \frac{n_2dV}{V-\rho_3} . \quad (99)$$

Integration then renders

$$t.\delta_2 = l_2\log_e(V-\rho_1) + m_2\log_e(V-\rho_2) + n_2\log_e(V-\rho_3) + C_2' \quad (100)$$

where $C_2'$ is the constant of integration.

Then $$t = \frac{l_2}{\delta_2}\log_e(V-\rho_1) + \frac{m_2}{\delta_2}\log_e(V-\rho_2) + \frac{n_2}{\delta_2}\log_e(V-\rho_3) + \frac{C_2'}{\delta_2}. \tag{101}$$

If $(V=V_1,\ t=t_1)$ is a point on the curve; that is, if

$$V = V_1 \quad when \quad t = t_1\ , \tag{102}$$

the time t at which the speed will attain some other value V is given by the equation,

$$\begin{aligned} t-t_1 = &\frac{l_2}{\delta_2}\log_e(V-\rho_1) + \frac{m_2}{\delta_2}\log_e(V-\rho_2) + \frac{n_2}{\delta_2}\log_e(V-\rho_3) \\ &- \frac{l_2}{\delta_2}\log_e(V_1-\rho_1) - \frac{m_2}{\delta_2}\log_e(V_1-\rho_2) - \frac{n_2}{\delta_2}\log_e(V_1-\rho_3)\ . \end{aligned} \tag{103}$$

Thus $$t = t_1 + \frac{1}{\delta_2}\left[l_2\log_e\frac{(V-\rho_1)}{(V_1-\rho_1)} + m_2\log_e\frac{(V-\rho_2)}{(V_1-\rho_2)} + n_2\log_e\frac{(V-\rho_3)}{(V_1-\rho_3)}\right]. \tag{104}$$

In common logarithms and expanded for convenience in computation,

$$t = t_1 + \frac{2.30}{\delta_2}\left[\begin{aligned} &l_2\log_{10}(V-\rho_1) + m_2\log_{10}(V-\rho_2) + n_2\log_{10}(V-\rho_3) \\ &- l_2\log_{10}(V_1-\rho_1) - m_2\log_{10}(V_1-\rho_2) - n_2\log_{10}(V_1-\rho_3)\end{aligned}\right]. \tag{105}$$

Combining equations (101) and (103) and letting

$$C_2 = \frac{C_2'}{\delta_2}\ , \tag{106}$$

$$C_2 = \frac{C_2'}{\delta_2} = t_1 - \frac{l_2}{\delta_2}\log_e(V_1-\rho_1) - \frac{m_2}{\delta_2}\log_e(V_1-\rho_2) - \frac{n_2}{\delta_2}\log_e(V_1-\rho_3) \tag{107}$$

or $$C_2 = t_1 - \frac{2.30}{\delta_2}\left[l_2\log_{10}(V_1-\rho_1) + m_2\log_{10}(V_1-\rho_2) + n_2\log_{10}(V_1-\rho_3)\right]. \tag{108}$$

Then, from equations (105) and (108),

$$t = C_2 + \frac{2.30}{\delta_2}\left[l_2\log_{10}(V-\rho_1) + m_2\log_{10}(V-\rho_2) + n_2\log_{10}(V-\rho_3)\right]. \tag{109}$$

## D - COASTING

In approaching stations, descending grades, slowing down for crossings, etc., it is generally advantageous from the standpoint of energy economy to shut off the electric power supply and allow the train to coast for a period before the brakes are applied. Under this condition, the motors do not supply any power to the train; in fact, they

draw a certain small amount from the kinetic energy of the train in order to overcome the friction and windage losses in the motors and gears. However, this draught is usually negligible, so it may be assumed that the tractive effort input to the train is zero during coasting. Also, by the above definition, the braking effort is zero during coasting. Hence, with

$$F = 0 \tag{110}$$

and

$$B = 0 \,, \tag{111}$$

the fundamental equation (1) reduces to

$$\frac{dV}{dt} = 0.01\left[-C-G-\frac{50}{\sqrt{T}}-0.03V-\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right] \tag{112}$$

The value of G will be positive or negative according as the train is going up-hill or down-hill. The magnitude and algebraic sign of the value of G will determine whether the acceleration, $\frac{dV}{dt}$, is positive, negative or zero. Expressed algebraically:

$$\frac{dV}{dt} < 0 \quad \text{if} \quad G > -\left[C+\frac{50}{\sqrt{T}}+0.03V+\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]\,; \tag{113}$$

$$\frac{dV}{dt} = 0 \quad \text{if} \quad G = -\left[C+\frac{50}{\sqrt{T}}+0.03V+\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]\,; \tag{114}$$

$$\frac{dV}{dt} > 0 \quad \text{if} \quad G < -\left[C+\frac{50}{\sqrt{T}}+0.03V+\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]. \tag{115}$$

In equation (112) let

$$\alpha_3 = 0.01\left(C+G+\frac{50}{\sqrt{T}}\right)\,, \tag{116}$$

$$\beta_3 = 0.0003 \quad , \tag{117}$$

$$\gamma_3 = \frac{0.00002X}{T}\left(1+\frac{N-1}{10}\right). \tag{118}$$

Then

$$\frac{dV}{dt} = -(\alpha_3+\beta_3V+\gamma_3V^2)\,, \tag{119}$$

$$-dt = \frac{dV}{\gamma_3V^2+\beta_3V+\alpha_3}\,, \tag{120}$$

$$-\gamma_3 dt = \frac{dV}{V^2+\frac{\beta_3}{\gamma_3}V+\frac{\alpha_3}{\gamma_3}}\,, \tag{121}$$

draw a certain small amount from the kinetic energy of the train in order to overcome the friction and windage losses in the motors and gears. However, this draught is usually negligible, so it may be assumed that the tractive effort input to the train is zero during coasting. Also, by the above definition, the braking effort is zero during coasting. Hence, with

$$F = 0 \tag{110}$$

and

$$B = 0 \, , \tag{111}$$

the fundamental equation (1) reduces to

$$\frac{dV}{dt} = 0.01\left[-C - G - \frac{50}{\sqrt{T}} - 0.03V - \frac{0.002X}{T}\left(1 + \frac{N-1}{10}\right)V^2\right] \tag{112}$$

The value of G will be positive or negative according as the train is going up-hill or down-hill. The magnitude and algebraic sign of the value of G will determine whether the acceleration, $\frac{dV}{dt}$, is positive, negative or zero. Expressed algebraically:

$$\frac{dV}{dt} < 0 \text{ if } G > -\left[C + \frac{50}{\sqrt{T}} + 0.03V + \frac{0.002X}{T}\left(1 + \frac{N-1}{10}\right)V^2\right] ; \tag{113}$$

$$\frac{dV}{dt} = 0 \text{ if } G = -\left[C + \frac{50}{\sqrt{T}} + 0.03V + \frac{0.002X}{T}\left(1 + \frac{N-1}{10}\right)V^2\right] ; \tag{114}$$

$$\frac{dV}{dt} > 0 \text{ if } G < -\left[C + \frac{50}{\sqrt{T}} + 0.03V + \frac{0.002X}{T}\left(1 + \frac{N-1}{10}\right)V^2\right] . \tag{115}$$

In equation (112) let

$$\alpha_3 = 0.01\left(C + G + \frac{50}{\sqrt{T}}\right) , \tag{116}$$

$$\beta_3 = 0.0003 \, , \tag{117}$$

$$\gamma_3 = \frac{0.00002X}{T}\left(1 + \frac{N-1}{10}\right) . \tag{118}$$

Then

$$\frac{dV}{dt} = -(\alpha_3 + \beta_3 V + \gamma_3 V^2) \, , \tag{119}$$

$$-dt = \frac{dV}{\gamma_3 V^2 + \beta_3 V + \alpha_3} \, , \tag{120}$$

$$-\gamma_3 dt = \frac{dV}{V^2 + \frac{\beta_3}{\gamma_3}V + \frac{\alpha_3}{\gamma_3}} \, , \tag{121}$$

$$-\gamma_3 dt = \frac{dV}{\left(V^2+\frac{\beta_3}{\gamma_3}V+\frac{\beta_3^2}{4\gamma_3^2}\right)+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)}\ , \qquad (122)$$

$$= \frac{dV}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)}\ . \qquad (123)$$

Now

$$d\left(V+\frac{\beta_3}{2\gamma_3}\right) = dV\ . \qquad (124)$$

Hence

$$-\gamma_3 dt = \frac{d\left(V+\frac{\beta_3}{2\gamma_3}\right)}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)}\ . \qquad (125)$$

There are three possible solutions of equation (125). The algebraic sign of $\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)$ determines which is the proper solution. The values of $\beta_3$ and $\gamma_3$ are positive as long as the train is moving. However, the value of $\alpha_3$ is positive or negative according as $G > -(C+\frac{50}{\sqrt{T}})$ or $G < -(C+\frac{50}{\sqrt{T}})$. Nevertheless, $\alpha_3$, $\beta_3$ and $\gamma_3$ are constants, fixed by track and operating conditions so it can readily be determined whether

$$\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} > 0\ , \quad \frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} = 0 \quad or \quad \frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} < 0\ .$$

Case I

If

$$\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} > 0\ , \qquad (126)$$

the solution of

$$-\gamma_3 dt = \frac{d\left(V+\frac{\beta_3}{2\gamma_3}\right)}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)} \qquad (125)$$

is

$$-\gamma_3 t = \frac{1}{\sqrt{\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}}}\ \cancel{\sin^{-1}}\left[\frac{V+\frac{\beta_3}{2\gamma_3}}{\sqrt{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)}}\right]+C_3'\ , \qquad (127)$$

or

$$t = \frac{-2}{\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\ \cancel{\sin^{-1}}\left[\frac{V+\frac{\beta_3}{2\gamma_3}}{\sqrt{V^2+\frac{\beta_3}{\gamma_3}V+\frac{\beta_3^2}{4\gamma_3^2}+\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}}}\right]-\frac{C_3'}{\gamma_3} \qquad (128)$$

The X er is in error
Sin$^{-1}$ is ok
Also tan$^{-1}$ ok if
$(V+\frac{\beta_3}{2\gamma_3})^2$ term
mitted in (127)

$$= \frac{-2}{\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\ \cancel{\sin^{-1}}\left[\frac{V+\frac{\beta_3}{2\gamma_3}}{\sqrt{\frac{1}{\gamma_3}(\gamma_3V^2+\beta_3V+\alpha_3)}}\right]-\frac{C_3'}{\gamma_3} \qquad (129)$$

$$= \frac{-2}{\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\ \cancel{\sin^{-1}}\left[\frac{2\gamma_3V+\beta_3}{\sqrt{4\gamma_3(\gamma_3V^2+\beta_3V+\alpha_3)}}\right]-\frac{C_3'}{\gamma_3}\ . \qquad (130)$$

$$= \ '' \quad \tan^{-1}\left[\frac{2\gamma_3V+\beta_3}{4\alpha_3\gamma_3-\beta_3^2}\right]-C_3''$$ ← new constant

$$-\gamma_3\,dt = \frac{dV}{\left(V^2+\frac{\beta_3}{\gamma_3}V+\frac{\beta_3^2}{4\gamma_3^2}\right)+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)}, \qquad (122)$$

$$= \frac{dV}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)}. \qquad (123)$$

Now

$$d\left(V+\frac{\beta_3}{2\gamma_3}\right) = dV . \qquad (124)$$

Hence

$$-\gamma_3\,dt = \frac{d\left(V+\frac{\beta_3}{2\gamma_3}\right)}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)} . \qquad (125)$$

There are three possible solutions of equation (125). The algebraic sign of $\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)$ determines which is the proper solution. The values of $\beta_3$ and $\gamma_3$ are positive as long as the train is moving. However, the value of $\alpha_3$ is positive or negative according as $G > -(C+\frac{20}{W})$ or $G < -(C+\frac{20}{W})$. Nevertheless, $\alpha_3, \beta_3$ and $\gamma_3$ are constants, fixed by track and operating conditions so it can readily be determined whether

$$\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} > 0 , \quad \frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} = 0 \quad \text{or} \quad \frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} < 0 .$$

Case I

If

$$\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2} > 0 , \qquad (126)$$

the solution of

$$-\gamma_3\,dt = \frac{d\left(V+\frac{\beta_3}{2\gamma_3}\right)}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)} \qquad (125)$$

is

$$-\gamma_3 t = \frac{1}{\sqrt{\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}}}\sin^{-1}\left[\frac{V+\frac{\beta_3}{2\gamma_3}}{\sqrt{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2+\left(\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}\right)}}\right] + C_3' , \qquad (127)$$

or

$$t = \frac{-2}{\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{V+\frac{\beta_3}{2\gamma_3}}{\sqrt{V^2+\frac{\beta_3}{\gamma_3}V+\frac{\beta_3^2}{4\gamma_3^2}+\frac{\alpha_3}{\gamma_3}-\frac{\beta_3^2}{4\gamma_3^2}}}\right] - \frac{C_3'}{\gamma_3} \qquad (128)$$

$$= \frac{-2}{\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{V+\frac{\beta_3}{2\gamma_3}}{\sqrt{\frac{1}{\gamma_3}\left(\gamma_3V^2+\beta_3V+\alpha_3\right)}}\right] - \frac{C_3'}{\gamma_3} \qquad (129)$$

$$= \frac{-2}{\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3V+\beta_3}{\sqrt{4\gamma_3\left(\gamma_3V^2+\beta_3V+\alpha_3\right)}}\right] - \frac{C_3'}{\gamma_3} . \qquad (130)$$

The x er is in error
$\sin^{-1}$ is give
Abs tan-1 ok
$\left(V+\frac{\beta_3}{2\gamma}\right)^2$ term
omitted in (127)

where time is expressed in seconds and the inverse sine ($\sin^{-1}$) in radians.

Since, in trigonometric tables, sines are compiled as functions of angles expressed in degrees, it is more convenient to modify the formula (130) so that the inverse sine can be derived directly from the tables in degrees and decimal fractions of a degree. This is accomplished by introducing the factor, 57.296, the number of degrees in a radian. Equation (130) then becomes

$$t = \frac{-2}{57.3\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3 V+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V^2+\beta_3 V+\alpha_3)}}\right]-\frac{C_3'}{\gamma_3} \qquad (131)$$

or

$$= \frac{-1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3 V+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V^2+\beta_3 V+\alpha_3)}}\right]-\frac{C_3'}{\gamma_3} \qquad (132)$$

where time is expressed in seconds and the inverse sine in degrees.

If ($t=t_3$ , $V=V_3$) be a point on the speed-time curve during the coasting period, that is, if

$$V = V_3 \quad \text{when} \quad t = t_3 \quad ; \qquad (133)$$

the time t, at which the train will attain any other speed V by coasting, is given by

$$t-t_3 = \frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3 V_3+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V_3^2+\beta_3 V_3+\alpha_3)}}\right]-\frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3 V+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V^2+\beta_3 V+\alpha_3)}}\right] \qquad (134)$$

or

$$t = t_3+\frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\left\{\sin^{-1}\left[\frac{2\gamma_3 V_3+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V_3^2+\beta_3 V_3+\alpha_3)}}\right]-\sin^{-1}\left[\frac{2\gamma_3 V+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V^2+\beta_3 V+\alpha_3)}}\right]\right\}. \qquad (135)$$

Combining equations (132) and (135), and letting

$$C_3 = -\frac{C_3'}{\gamma_3} \quad , \qquad (136)$$

$$C_3 = t_3+\frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3 V_3+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V_3^2+\beta_3 V_3+\alpha_3)}}\right] , \qquad (137)$$

and

$$t = C_3-\frac{1}{28.6\sqrt{4\alpha_3\gamma_3+\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3 V+\beta_3}{\sqrt{4\gamma_3(\gamma_3 V^2+\beta_3 V+\alpha_3)}}\right] . \qquad (138)$$

where time is expressed in seconds and the inverse sine ($\sin^{-1}$) in radians.

Since, in trigonometric tables, sines are compiled as functions of angles expressed in degrees, it is more convenient to modify the formula (130) so that the inverse sine can be derived directly from the tables in degrees and decimal fractions of a degree. This is accomplished by introducing the factor, 57.296, the number of degrees in a radian. Equation (130) then becomes

$$t = \frac{-2}{57.3\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3V+\beta_3}{\sqrt{4\gamma_3(\gamma_3V^2+\beta_3V+\alpha_3)}}\right]-\frac{C_3'}{\gamma_3} \qquad (131)$$

or

$$= \frac{-1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3V+\beta_3}{\sqrt{4\gamma_3(\gamma_3V^2+\beta_3V+\alpha_3)}}\right]-\frac{C_3'}{\gamma_3} \qquad (132)$$

where time is expressed in seconds and the inverse sine in degrees.

If ($t=t_3$ , $V=V_3$) be a point on the speed-time curve during the coasting period, that is, if

$$V = V_3 \text{ when } t = t_3 \ ; \qquad (133)$$

the time $t$, at which the train will attain any other speed $V$ by coasting, is given by

$$t - t_3 = \frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3V_3+\beta_3}{\sqrt{4\gamma_3(\gamma_3V_3^2+\beta_3V_3+\alpha_3)}}\right]-\frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3V+\beta_3}{\sqrt{4\gamma_3(\gamma_3V^2+\beta_3V+\alpha_3)}}\right] \qquad (134)$$

$$\text{or } t = t_3 + \frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\left\{\sin^{-1}\left[\frac{2\gamma_3V_3+\beta_3}{\sqrt{4\gamma_3(\gamma_3V_3^2+\beta_3V_3+\alpha_3)}}\right]-\sin^{-1}\left[\frac{2\gamma_3V+\beta_3}{\sqrt{4\gamma_3(\gamma_3V^2+\beta_3V+\alpha_3)}}\right]\right\}. \qquad (135)$$

Combining equations (132) and (135), and letting

$$C_3 = -\frac{C_3'}{\gamma_3} \ , \qquad (136)$$

$$C_3 = t_3 + \frac{1}{28.6\sqrt{4\alpha_3\gamma_3-\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3V_3+\beta_3}{\sqrt{4\gamma_3(\gamma_3V_3^2+\beta_3V_3+\alpha_3)}}\right] . \qquad (137)$$

and

$$t = C_3 - \frac{1}{28.6\sqrt{4\alpha_3\gamma_3+\beta_3^2}}\sin^{-1}\left[\frac{2\gamma_3V+\beta_3}{-\sqrt{4\gamma_3(\gamma_3V^2+\beta_3V+\alpha_3)}}\right] . \qquad (138)$$

Case II

If
$$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} = 0 \ , \tag{139}$$

equation (125) becomes
$$\gamma_3 dt = \frac{-d(V + \frac{\beta_3}{2\gamma_3})}{(V + \frac{\beta_3}{2\gamma_3})^2} \ . \tag{140}$$

The solution of this is
$$\gamma_3 t = \frac{1}{V + \frac{\beta_3}{2\gamma_3}} + C_3'' \tag{141}$$

or
$$t = \frac{2}{2\gamma_3 V + \beta_3} + \frac{C_3''}{\gamma_3} \ . \tag{142}$$

If $V = V_3$ when $t = t_3$ , (133)

the time t, at which the train will reach any other speed V by coasting, is given by
$$t - t_3 = \frac{2}{2\gamma_3 V + \beta_3} - \frac{2}{2\gamma_3 V_3 + \beta_3} \tag{143}$$

or
$$t = t_3 + \frac{2}{2\gamma_3 V + \beta_3} - \frac{2}{2\gamma_3 V_3 + \beta_3} \ . \tag{144}$$

Combining equations (142) and (144), and letting
$$C_3 = \frac{C_3''}{\gamma_3} \ , \tag{145}$$

$$C_3 = t_3 - \frac{2}{2\gamma_3 V_3 + \beta_3} \ . \tag{146}$$

Then
$$t = C_3 + \frac{2}{2\gamma_3 V + \beta_3} \ . \tag{147}$$

Case III

If
$$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} < 0 \ , \tag{148}$$

$$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} = -\left|\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2}\right| \tag{149}$$

where the parallel vertical lines signify that the absolute, or numerical, value or the quantity, that they enclose, is taken with

Case II

If $$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} = 0 \quad , \qquad (139)$$

equation (125) becomes

$$\gamma_3\, dt = \frac{-d\left(V+\frac{\beta_3}{2\gamma_3}\right)}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2} \quad . \qquad (140)$$

The solution of this is

$$\gamma_3 t = \frac{1}{V+\frac{\beta_3}{2\gamma_3}} + C_3'' \qquad (141)$$

or

$$t = \frac{2}{2\gamma_3 V+\beta_3} + \frac{C_3''}{\gamma_3} \quad . \qquad (142)$$

If $V = V_3$ when $t = t_3$ , (132)

the time $t$, at which the train will reach any other speed $V$ by coasting, is given by

$$t - t_3 = \frac{2}{2\gamma_3 V+\beta_3} - \frac{2}{2\gamma_3 V_3+\beta_3} \qquad (143)$$

or

$$t = t_3 + \frac{2}{2\gamma_3 V+\beta_3} - \frac{2}{2\gamma_3 V_3+\beta_3} \quad . \qquad (144)$$

Combining equations (142) and (144), and letting

$$C_3 = \frac{C_3''}{\gamma_3} \quad , \qquad (145)$$

$$C_3 = t_3 - \frac{2}{2\gamma_3 V_3+\beta_3} \quad . \qquad (146)$$

Then

$$t = C_3 + \frac{2}{2\gamma_3 V+\beta_3} \quad . \qquad (147)$$

Case III

If $$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} < 0 \quad , \qquad (148)$$

$$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} = -\left|\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2}\right| \qquad (149)$$

where the parallel vertical lines signify that the absolute, or numerical, value of the quantity, that they enclose, is taken with

the positive algebraic sign.

Substituting the relation (149) into equation (125), gives

$$-\gamma_3 dt = \frac{d\left(V+\frac{\beta_3}{2\gamma_3}\right)}{\left(V+\frac{\beta_3}{2\gamma_3}\right)^2 - \left|\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2}\right|} \tag{150}$$

or

$$\gamma_3 dt = \frac{d\left(V+\frac{\beta_3}{2\gamma_3}\right)}{\left|\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2}\right| - \left(V+\frac{\beta_3}{2\gamma_3}\right)^2} . \tag{151}$$

The solution of this is

$$\gamma_3 t = \frac{1}{2\sqrt{\left|\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2}\right|}} \log_e \left[\frac{\sqrt{\left|\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2}\right|} + \left(V+\frac{\beta_3}{2\gamma_3}\right)}{\sqrt{\left|\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2}\right|} - \left(V+\frac{\beta_3}{2\gamma_3}\right)}\right] + C_3''' . \tag{152}$$

Then

$$t = \frac{1}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|}} \log_e \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V}\right] + \frac{C_3'''}{\gamma_3} . \tag{153}$$

In common logarithms, in order to facilitate computation,

$$t = \frac{2.30}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|}} \log_{10} \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V}\right] + \frac{C_3'''}{\gamma_3} . \tag{154}$$

If $\quad V = V_3 \quad$ when $\quad t = t_3$ , (133)

the time t, at which the train will attain any other speed V by coasting, is given by

$$t - t_3 = \frac{2.30}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|}} \left\{\log_{10} \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V}\right] - \log_{10} \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V_3}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V_3}\right]\right\}, \tag{155}$$

so that

$$t = t_3 + \frac{2.30}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|}} \left\{\log_{10} \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V}\right] - \log_{10} \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V_3}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V_3}\right]\right\}. \tag{156}$$

Combining equations (154) and (156), and letting

$$C_3 = \frac{C_3'''}{\gamma_3} , \tag{157}$$

$$C_3 = t_3 - \frac{2.30}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|}} \log_{10} \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V_3}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V_3}\right] , \tag{158}$$

and

$$t = C_3 + \frac{2.30}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|}} \log_{10} \left[\frac{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} + \beta_3 + 2\gamma_3 V}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|} - \beta_3 - 2\gamma_3 V}\right] . \tag{159}$$

## E - BRAKING

When the brakes are applied to a train, the power supply is usually shut off as in coasting, so the input tractive effort, F in equation (1) is zero during the braking period. Hence, equation (1) becomes

$$\frac{dV}{dt} = 0.01\left[-B-C-G-\frac{50}{\sqrt{T}}-0.03V-\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]. \qquad (160)$$

Brakes are most commonly applied for the purpose of retarding the motion of a train. However, occasionally, as during the descent of grades, the brakes are partially applied and the speed allowed to increase though not to such an extent as it would if the brakes were not applied. Therefore, in order to be general, the formulae for speed-time relations during braking must provide for positive, zero and negative acceleration. Expressed algebraically:

$$\frac{dV}{dt} < 0 \quad \text{if} \quad G > -\left[B+C+\frac{50}{\sqrt{T}}+0.03V+\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]; \qquad (161)$$

$$\frac{dV}{dt} = 0 \quad \text{if} \quad G = -\left[B+C+\frac{50}{\sqrt{T}}+0.03V+\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]; \qquad (162)$$

$$\frac{dV}{dt} > 0 \quad \text{if} \quad G < -\left[B+C+\frac{50}{\sqrt{T}}+0.03V+\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]. \qquad (163)$$

The acceleration is

$$A = \frac{dV}{dt} = 0.01\left[-B-C-G-\frac{50}{\sqrt{T}}-0.03V-\frac{0.002X}{T}\left(1+\frac{N-1}{10}\right)V^2\right]. \qquad (164)$$

Let

$$\alpha_4 = 0.01\left(B+C+G+\frac{50}{\sqrt{T}}\right), \qquad (165)$$

$$\beta_4 = 0.0003, \qquad (166)$$

$$\gamma_4 = \frac{0.00002X}{T}\left(1+\frac{N-1}{10}\right). \qquad (167)$$

Then

$$\frac{dV}{dt} = -(\alpha_4+\beta_4V+\gamma_4V^2). \qquad (168)$$

A comparison of equations (116) to (119) and (165) to (168) shows that the speed-time relations during the coasting period are different from those in the braking period only in that $B \neq 0$ in the latter; that is

$$\alpha_4 = 0.01\left(B + C + G + \frac{50}{\sqrt{T}}\right) \tag{165}$$

while

$$\alpha_3 = 0.01\left(C + G + \frac{50}{\sqrt{T}}\right) \ . \tag{116}$$

Hence, since and are both constants, the solutions for the two periods will be similar in form and the resulting formulae for the braking period can be written directly.

Thus, if $(t=t_4\ ,\ V=V_4)$ be a point on the speed-time curve in the braking period, that is, if

$$V = V_4 \quad \text{when} \quad t = t_4 \ , \tag{169}$$

the time t, at which the speed will attain some other value V under constant application of the brakes, is given by the following formulae:

Case I

If

$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2} > 0 \ , \tag{170}$$

$$t = t_4 + \frac{1}{28.6\sqrt{4\alpha_4\gamma_4 - \beta_4^2}}\left\{\sin^{-1}\left[\frac{2\gamma_4 V_4 + \beta_4}{\sqrt{4\gamma_4(\gamma_4 V_4^2 + \beta_4 V_4 + \alpha_4)}}\right] - \sin^{-1}\left[\frac{2\gamma_4 V + \beta_4}{\sqrt{4\gamma_4(\gamma_4 V^2 + \beta_4 V + \alpha_4)}}\right]\right\} . \tag{171}$$

Letting

$$C_4 = t_4 + \frac{1}{28.6\sqrt{4\alpha_4\gamma_4 - \beta_4^2}}\sin^{-1}\left[\frac{2\gamma_4 V_4 + \beta_4}{\sqrt{4\gamma_4(\gamma_4 V_4^2 + \beta_4 V_4 + \alpha_4)}}\right] \ , \tag{172}$$

$$t = C_4 - \frac{1}{28.6\sqrt{4\alpha_4\gamma_4 - \beta_4^2}}\sin^{-1}\left[\frac{2\gamma_4 V + \beta_4}{\sqrt{4\gamma_4(\gamma_4 V^2 + \beta_4 V + \alpha_4)}}\right] . \tag{173}$$

In these formulae, time is expressed in seconds, speed in miles per hour, and inverse sines in degrees.

Case II

If
$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2} = 0 \quad , \tag{174}$$

$$t = t_4 + \frac{2}{2\gamma_4 V + \beta_4} - \frac{2}{2\gamma_4 V_4 + \beta_4} \; . \tag{175}$$

Letting
$$C_4 = t_4 - \frac{2}{2\gamma_4 V_4 + \beta_4} \; , \tag{176}$$

$$t = C_4 + \frac{2}{2\gamma_4 V + \beta_4} \; . \tag{177}$$

Case III

If
$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2} < 0 \quad , \tag{178}$$

so that
$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2} = -\left|\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2}\right| \quad , \tag{179}$$

$$t = t_4 + \frac{2.30}{\sqrt{|4\alpha_4\gamma_4 - \beta_4^2|}}\left\{\log_{10}\left[\frac{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}+\beta_4+2\gamma_4 V}{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}-\beta_4-2\gamma_4 V}\right] - \log_{10}\left[\frac{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}+\beta_4+2\gamma_4 V_4}{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}-\beta_4-2\gamma_4 V_4}\right]\right\} . \tag{180}$$

Letting
$$C_4 = t_4 - \frac{2.30}{\sqrt{|4\alpha_4\gamma_4 - \beta_4^2|}}\log_{10}\left[\frac{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}+\beta_4+2\gamma_4 V_4}{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}-\beta_4-2\gamma_4 V_4}\right] \; , \tag{181}$$

$$t = C_4 + \frac{2.30}{\sqrt{|4\alpha_4\gamma_4 - \beta_4^2|}}\log_{10}\left[\frac{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}+\beta_4+2\gamma_4 V}{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}-\beta_4-2\gamma_4 V}\right] . \tag{182}$$

Special Case, $V = 0$

In stopping a train by the application of friction brakes, if the speed is $V_4$ at the instant $t_4$ in the braking period, the time $t$, at which the train will stop, is given by letting $V = 0$ in equation (171). Then

$$t = t_4 + \frac{1}{28.6\sqrt{4\alpha_4\gamma_4 - \beta_4}}\left\{\sin^{-1}\left[\frac{2\gamma_4 V_4 + \beta_4}{\sqrt{4\gamma_4(\gamma_4 V_4^2 + \beta_4 V + \alpha_4)}}\right] - \sin^{-1}\left[\frac{\beta_4}{\sqrt{4\alpha_4\gamma_4}}\right]\right\} \tag{183}$$

## F - SUMMARY OF PRINCIPAL FORMULAE

In conclusion, the principal formulae are grouped together in order to facilitate reference to them.

The acceleration formula , or fundamental differential equation of train speed is

$$\frac{dV}{dt} = 0.01\left[F - B - C - G - \frac{50}{\sqrt{T}} - 0.03V - \frac{0.002X}{T}\left(1 + \frac{N-1}{10}\right)V^2\right]. \qquad (1)$$

The formula for the tractive effort input to the train, when rated voltage is applied to the motor terminals, is

$$F = \frac{h_1}{V - h_2}. \qquad (22)$$

For the starting period, that is, with constant tractive effort input to the train,

$$t = t_o + \frac{2.30}{\sqrt{4\alpha_1\gamma_1 + \beta_1^2}}\left[\begin{array}{l} \log_{10}(\sqrt{4\alpha_1\gamma_1+\beta_1^2} + \beta_1 + 2\gamma_1 V) - \log_{10}(\sqrt{4\alpha_1\gamma_1+\beta_1^2} - \beta_1 - 2\gamma_1 V) \\ -\log_{10}(\sqrt{4\alpha_1\gamma_1+\beta_1^2} + \beta_1 + 2\gamma_1 V_o) + \log_{10}(\sqrt{4\alpha_1\gamma_1+\beta_1^2} - \beta_1 - 2\gamma_1 V_o) \end{array}\right]. \qquad (39)$$

If $V_o = 0$ and $t_o = 0$

$$t_s = \frac{2.30}{\sqrt{4\alpha_1\gamma_1+\beta_1^2}}\left[\log_{10}(2\alpha_1 - \beta_1 V + V\sqrt{4\alpha_1\gamma_1+\beta_1^2}) - \log_{10}(2\alpha_1 - \beta_1 V - V\sqrt{4\alpha_1\gamma_1+\beta_1^2})\right]. \qquad (47)$$

During acceleration with rated voltage applied to the motors,

$$t = t_1 + \frac{1}{\delta_2}\left[l_2 \log_e \frac{V-\rho_1}{V_1-\rho_1} + m_2 \log_e \frac{V-\rho_2}{V_1-\rho_2} + n_2 \log_e \frac{V-\rho_3}{V_1-\rho_3}\right]. \qquad (104)$$

During coasting,

$$t = t_3 + \frac{1}{28.6\sqrt{4\alpha_3\gamma_3 - \beta_3^2}}\left\{\sin^{-1}\left[\frac{2\gamma_3 V_3 + \beta_3}{\sqrt{4\gamma_3(\gamma_3 V_3^2 + \beta_3 V_3 + \alpha_3)}}\right] - \sin^{-1}\left[\frac{2\gamma_3 V + \beta_3}{\sqrt{4\gamma_3(\gamma_3 V^2 + \beta_3 V + \alpha_3)}}\right]\right\} \qquad (135)$$

provided

$$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} > 0 \;; \qquad (126)$$

$$t = t_3 + \frac{2}{2\gamma_3 V + \beta_3} - \frac{2}{2\gamma_3 V_3 + \beta_3} \qquad (144)$$

if

$$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3} = 0 \;; \qquad (139)$$

and

$$t_3 = t_3 + \frac{2.30}{\sqrt{|4\alpha_3\gamma_3 - \beta_3^2|}}\left[\begin{array}{l}\log_{10}\left(\sqrt{|4\alpha_3\gamma_3-\beta_3^2|}+\beta_3+2\gamma_3 V\right) - \log_{10}\left(\sqrt{|4\alpha_3\gamma_3-\beta_3^2|}-\beta_3-2\gamma_3 V\right) \\ -\log_{10}\left(\sqrt{|4\alpha_3\gamma_3-\beta_3^2|}+\beta_3+2\gamma_3 V_3\right) + \log_{10}\sqrt{|4\alpha_3\gamma_3-\beta_3^2|}-\beta_3-2\gamma_3 V_3)\end{array}\right] \qquad (156)$$

if

$$\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} < 0 \quad . \qquad (148)$$

During braking,

$$t = t_4 + \frac{1}{28.6\sqrt{4\alpha_4\gamma_4-\beta_4^2}}\left\{\sin^{-1}\left[\frac{2\gamma_4 V_4+\beta_4}{\sqrt{4\gamma_4(\gamma_4 V_4^2+\beta_4 V_4+\alpha_4)}}\right] - \sin^{-1}\left[\frac{2\gamma_4 V+\beta_4}{\sqrt{4\gamma_4(\gamma_4 V^2+\beta_4 V+\alpha_4)}}\right]\right\} \qquad (171)$$

provided

$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2} > 0 \quad ; \qquad (170)$$

$$t = t_4 + \frac{2}{2\gamma_4 V+\beta_4} - \frac{2}{2\gamma_4 V+\beta_4} \qquad (175)$$

if

$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2} = 0 \quad ; \qquad (174)$$

and

$$t = t_4 + \frac{2.30}{\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}}\left[\begin{array}{l}\log_{10}\left(\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}+\beta_4+2\gamma_4 V\right) - \log_{10}\left(\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}-\beta_4-2\gamma_4 V\right) \\ -\log_{10}\left(\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}+\beta_4+2\gamma_4 V_4\right) + \log_{10}\left(\sqrt{|4\alpha_4\gamma_4-\beta_4^2|}-\beta_4-2\gamma_4 V_4\right)\end{array}\right] \qquad (180)$$

if

$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{2\gamma_4^2} < 0 \quad . \qquad (178)$$

If the speed steadily decreases to $V = 0$ under the application of the brakes, the train will stop when

$$t = t_4 + \frac{1}{28.6\sqrt{4\alpha_4\gamma_4-\beta_4^2}}\left\{\sin^{-1}\left[\frac{2\gamma_4 V_4+\beta_4}{\sqrt{4\gamma_4(\gamma_4 V_4^2+\beta_4 V_4+\alpha_4)}}\right] - \sin^{-1}\left[\frac{\beta_4}{\sqrt{4\alpha_4\gamma_4}}\right]\right\} \quad . \qquad (183)$$

IV

## EXAMPLES

These examples are given for the purpose of illustrating the methods of applying the foregoing formulae. For simplicity, they are all computed for the same train.

Train Data

| | |
|---|---|
| Number of cars, | N = 6 |
| Motor cars, | 3 |
| Trailers, | 3 |
| Motors per motor car, | 2 |
| Number of motors, | M = 6 |
| Weight of three motor cars, | 3 X 28 = 84 tons |
| Weight of three trailers, | 3 X 22 = 66 tons |
| Total weight of train, | T = 150 tons |
| Gross train weight per motor | 150 ÷ 6 = 25 tons |
| Area of projected cross-section | X = 110 sq.ft. |
| Average starting current per motor | 350 amperes. |
| Motor Characteristics | Fig. 2 |

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1 - Starting

How much time will be consumed in the starting period on level tangent track, that is, to accelerate the train from standstill to 17.2 miles per hour, the speed, on the motors' normal speed curve, corresponding to the average starting current of 350 amperes ?

From the characteristic curves, Fig. 2, the tractive effort, corresponding to 350 amperes, is 5000 lbs. per motor. Hence, the tractive effort per ton of gross train weight is

$$F = 5000/25 = 200 \text{ lbs. per ton.}$$

Since the track is level and straight, and the brakes are not applied during starting, $B = C = G = 0$ .

Substituting in equations (24), (25) and (26),

$$\alpha_1 = 0.01\left(200 - \frac{50}{\sqrt{150}}\right)$$

$$= 1.96 \; ,$$

$$\beta_1 = 0.0003 \qquad 0.3\times10^{-3} \; ,$$

$$\gamma_1 = \frac{0.00002}{150}\, 110\left(1 + \frac{6-1}{10}\right)$$

$$= 22\times10^{-6} \; .$$

Then formula (47) gives

$$t_s = \frac{2.30}{\sqrt{4\times1.96\times22\times10^{-6} + 0.09\times10^{-6}}}\left[\begin{array}{l}\log_{10}\left(2\times1.96 - 0.0003\times17.2 + 17.2\sqrt{4\times1.96\times22\times10^{-6} + 0.09\times10^{-6}}\right)\\ -\log_{10}\left(2\times1.96 - 0.0003\times17.2 - 17.2\sqrt{4\times1.96\times22\times10^{-6} + 0.09\times10^{-6}}\right)\end{array}\right]$$

$$= 8.8 \text{ seconds} \; .$$

2 - Balancing Speed

What will be the balancing speed on a one per cent up-grade ?

On the straight line AA, which approximates the motor power output current curve, Fig. 2,

$$P' = 160 \quad \text{when} \quad I = 325$$

and $$P' = 49.4 \quad \text{when} \quad I = 100.$$

Substituting these values in equation (2) gives

$$160 = h_1' + 325\, h_2'$$

and $$49.4 = h_1' + 100\, h_2' \ .$$

Solving these as simultaneous equations,

$$110.6 = 225\, h_2' \ ,$$

$$h_2' = 0.4915$$

and $$h_1' = 0.250 \ .$$

On the straight line BB, which approximates the tractive effort-current curve, Fig. 2,

$$F' = 4980 \text{ when } I = 350$$

and $$F' = 880 \text{ when } I = 100 \ .$$

Substituting these values in equation (10) gives

$$4980 = h_3 + 350\, h_4$$

and $$880 = h_3 + 100\, h_4 \ .$$

Then $$4100 = 250\, h_4 \ ,$$

$$h_4 = 16.4$$

and $$h_3 = -760 \ .$$

Substituting these values of $h_1'$ , $h_2'$ , $h_3$ and $h_4$ in equations (20) and (21) gives

$$h_1 = \frac{503 \text{ X } 6}{150}\left( 0.25 + \frac{760 \text{ X } 0.4915}{16.4}\right)$$

$$= 463.4$$

and

$$h_2 = 503 \left(\frac{0.4915}{16.4}\right)$$

$$= 15.0 \quad .$$

Thus equation (22) gives

$$F = \frac{463.4}{V - 15.0} \quad .$$

For one per cent grade,

$$G = 2000 \sin( \tan^{-1} 0.01 )$$

$$= 20.0 \ .$$

Equations (50), (52), (53) and (54) give

$$\delta_2 = 0.01 \left[\frac{-0.002}{150} 110\left(1 + \frac{6-1}{10}\right)\right]$$

$$= -22 \times 10^{-6} \ ,$$

$$\alpha_2 = -\frac{0.01}{22 \times 10^{-6}}\left[463.4 + 15.0\left(20.0 + \frac{50}{\sqrt{150}}\right)\right]$$

$$= -374.8 \times 10^3 \ ,$$

$$\beta_2 = -\frac{10^4}{22}\left[0.03 \times 15.0 - \left(20.0 + \frac{50}{\sqrt{150}}\right)\right]$$

$$= 10.74 \times 10^3 \ ,$$

$$\gamma_4 = \frac{-10^4}{22}\left[\frac{0.002 \times 110 \times 15.0}{150}\left(1 + \frac{6-1}{10}\right) - 0.03\right]$$

$$= -1.364 \ .$$

Then equations (61) and (62) give

$$q = 10740 - \frac{(1.364)^2}{3}$$

$$= 10.74 \text{ X } 10^3$$

and

$$r = -374800 + \frac{10740 \text{ X } 1.364}{3} - \frac{2(1.364)^3}{27}$$

$$= -369.9 \text{ X } 10^3 \ .$$

Relations (74) and (75) give

$$m = \left[\frac{369900}{2} + \sqrt{\frac{(369900)^2}{4} + \frac{(10740)^3}{27}}\right]^{1/3}$$
$$= 74.17 \quad ,$$

and

$$n = \left[\frac{369900}{2} - \sqrt{\frac{(369900)^2}{4} + \frac{(10740)^3}{27}}\right]^{1/3}$$
$$= -46.11 \quad .$$

Hence, by formulae (78), the balancing speed is

$$\rho_1 = \frac{1.364}{3} + 74.17 - 46.11$$
$$= 28.52 \text{ miles per hour.}$$

3 - Accelerating

If the train is accelerated from the close of the starting period on a one per cent grade, how long after starting from standstill will the speed become 28.0 miles per hour ?

From the preceding examples and equations (76), (78), (79) and (80),

$$t_1 = 8.8 \quad ,$$
$$V_1 = 17.2 \quad ,$$
$$V = 28.0 \quad ,$$
$$\delta_2 = -22 \times 10^{-6} ,$$
$$\rho_1 = 28.52 \quad ,$$
$$\rho_2 = \frac{1.364}{3} + 74.17\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) - 46.11\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$
$$= -13.58 + j24.30 \quad ,$$
$$\rho_3 = \frac{1.364}{3} + 74.17\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) - 46.11\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$
$$= -13.58 - j24.30 \quad .$$

From equations (96), (97) and (98),

$$m_2 = \frac{-13.58 + j24.30 - 15.0}{3(-13.58+j24.30)^2 + 2(-1.364)(-13.58+j24.30) + 10740}$$
$$= -0.00338 + j0.00182 \quad ,$$

$$m = \left[\frac{369900}{2} + \sqrt{\frac{(369900)^2}{4} + \frac{(10740)^3}{27}}\right]^{1/3}$$
$$= 74.17 \,,$$
and
$$n = \left[\frac{369900}{2} - \sqrt{\frac{(369900)^2}{4} + \frac{(10740)^3}{27}}\right]^{1/3}$$
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$$\rho_1 = 28.52 \,,$$
$$\rho_2 = \frac{1.364}{3} + 74.17\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) - 46.11\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)$$
$$= -13.58+j24.30 \,,$$
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$$= -0.00338+j0.00182 \,,$$

$$l_2 = \frac{28.52 - 15.0}{3(28.52)^2 + 2(-1.364)(28.52) + 10740}$$

$$= 0.00103 \; ,$$

$$n_2 = \frac{-13.58 - j24.30 - 15.0}{3(-13.58 - j24.30)^2 + 2(-1.364)(-13.58 - j24.30) + 10740}$$

$$= -0.00338 - j0.00182 \; .$$

Substituting these values in equation (104)

$$t = 8.8 + \frac{1}{-22 \times 10^{-6}} \left[ 0.00103 \log_e \frac{(28.00 - 28.52)}{(17.2 - 28.52)} + (-0.00338 + j0.00182) \log_e \frac{(28.00 + 13.58 - j24.30)}{(17.2 + 13.58 - j24.30)} + (-0.00338 - j0.00182) \log_e \frac{(28.00 + 13.58 + j24.30)}{(17.2 + 13.58 + j24.30)} \right]$$

$$t = 8.8 - \frac{10^6}{22} \left[ 0.00103 \log_e \left(\frac{0.52}{11.32}\right) + (-0.00338 + j0.00182) \log_e (41.58 - j24.30) - (-0.00338 + j0.00182) \log_e (30.78 - j24.30) + (-0.00338 - j0.00182) \log_e (41.58 + j24.30) - (-0.00338 - j0.00182) \log_e (30.78 + j24.30) \right]$$

For logarithms of the complex quantities, in general

$$\log_e (x + jy) = \log_e |x + jy| + j \tan^{-1} \frac{y}{x} + 2jm\pi , \qquad m = \text{integer},$$

$$= \log_e (+\sqrt{x^2 + y^2}) + j \tan^{-1} \frac{y}{x} + 2jm\pi , \qquad j = +\sqrt{-1} ,$$

$$= 2.30 \log_{10} (\sqrt{x^2 + y^2}) + j \tan^{-1} \frac{y}{x} + 2jm\pi \; .$$

However, since m is any integer, this formula gives an infinite number of values of $\log_e (x + jy)$ . But only ~~the principal~~ real values of the time ~~definite integrals~~ are concerned in speed-time determinations, so the last term, $2jm\pi$ , may be neglected; that is, let $m = 0$ . The resulting solution then is

$$t = 8.8 + 230.7 = 239.5 \text{ seconds} ,$$

approximately 4.00 minutes .

$$\eta_1 = \frac{28.52 - 15.0}{3(28.52)^2 + 2(-1.364)(28.52) + 10740}$$

$$= 0.00103 ,$$

$$\eta_2 = \frac{-13.58 - j24.30 - 15.0}{3(-13.58 - j24.30)^2 + 2(-1.364)(-13.58 - j24.30) + 10740}$$

$$= -0.00338 - j0.00182 .$$

Substituting these values in equation (104)

$$t = 8.8 + \frac{1}{-22 \times 10^{-6}} \left[ 0.00103 \log_e \frac{(28.00 - 28.52)}{(17.2 - 28.52)} + (-0.00338 + j0.00182) \log_e \frac{(28.00 + 13.58 - j24.30)}{(17.2 + 13.58 - j24.30)} + (-0.00338 - j0.00182) \log_e \frac{(28.00 + 13.58 + j24.30)}{(17.2 + 13.58 + j24.30)} \right]$$

$$t = 8.8 - \frac{10^6}{22} \left[ 0.00103 \log_e \left(\frac{0.52}{11.32}\right) + (-0.00338 + j0.00182) \log_e (41.58 - j24.30) - (-0.00338 + j0.00182) \log_e (30.78 - j24.30) + (-0.00338 - j0.00182) \log_e (41.58 + j24.30) - (-0.00338 - j0.00182) \log_e (30.78 + j24.30) \right]$$

For logarithms of the complex quantities, in general

$$\log_e(x + jy) = \log_e |x + jy| + j \tan^{-1} \frac{y}{x} + 2jm\pi , \qquad m = \text{integer},$$

$$= \log_e \left(+\sqrt{x^2 + y^2}\right) + j \tan^{-1} \frac{y}{x} + 2jm\pi , \quad j = +\sqrt{-1} .$$

$$= 2.30 \log_{10} \left(\sqrt{x^2 + y^2}\right) + j \tan^{-1} \frac{y}{x} + 2jm\pi .$$

However, since m is any integer, this formula gives an infinite number of values of $\log_e(x+jy)$. But only the principal values of definite integrals are concerned in speed-time determinations, so the last term, $2jm\pi$, may be neglected; that is let $m = 0$. The resulting solution then is

$$t = 8.8 + 230.7 = 239.5 \text{ seconds},$$

approximately 4.00 minutes.

## 4 - Coasting

If the power is shut off when the speed reaches 28.0 miles per hour, and the train is allowed to coast up the one per cent grade, how long after the train started will the speed become 10.0 miles per hour ?

Equations (116), (117) and (118) give

$$
\begin{aligned}
\alpha_3 &= 0.01\left(20.0 + \frac{50}{\sqrt{150}}\right) \\
&= 0.2408 \ , \\
\beta_3 &= 0.0003 \ , \\
\gamma_3 &= \frac{0.00002}{150}\,110\left(1 + \frac{6-1}{10}\right) \\
&= 22 \times 10^{-6} \ ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\alpha_3}{\gamma_3} - \frac{\beta_3^2}{4\gamma_3^2} &= \frac{0.2408}{22 \times 10^{-6}} - \frac{0.09 \times 10^{-6}}{4 \times 484 \times 10^{-6}} \\
&= 10.9 \times 10^{3} > 0 \ .
\end{aligned}
$$

Therefore, formula (135) applies and

$$
\begin{aligned}
t &= 239.5 + \frac{1}{28.6\sqrt{4 \times 0.2408 \times 22 \times 10^{-6} - 0.09 \times 10^{-6}}}\left[\sin^{-1}\frac{2 \times 22 \times 10^{-6} \times 28.0 + 0.0003}{\sqrt{4 \times 22 \times 10^{-6}\left(22 \times 10^{-6} \times \overline{28.0}^{2} + 0.0003 \times 28.0 + 0.2408\right)}} - \sin^{-1}\frac{2 \times 22 \times 10^{-6} \times 10 + 0.0003}{\sqrt{4 \times 22 \times 10^{-6}\left(22 \times 10^{-4} + 0.0003 \times 10.0 + 0.2408\right)}}\right] \\
&= 239.5 + 70.6 \ . \\
&= 310 \text{ seconds} \ , \\
&\doteq 5.17 \text{ minutes} \ .
\end{aligned}
$$

5 - Braking

From the time the speed reaches 10.0 miles per hour until the train stops, an average braking effort of 200 pounds per ton of gross train weight is applied by friction brakes, the grade continuing at + 1.00 per cent. How much time will have been consumed in making the run from start to stop ?

From equations (165), (166) and (167),

$$\alpha_4 = 0.01\left(200 + 20.0 + \frac{50}{\sqrt{150}}\right)$$

$$= 2.2408 ,$$

$$\beta_4 = 0.0003 ,$$

$$\gamma_4 = \frac{0.00002}{150} 110\left(1 + \frac{6-1}{10}\right)$$

$$= 22 \times 10^{-6} ,$$

$$\frac{\alpha_4}{\gamma_4} - \frac{\beta_4^2}{4\gamma_4^2} = \frac{2.2408}{22 \times 10^{-6}} - \frac{0.09 \times 10^{-6}}{4 \times 484 \times 10^{-12}}$$

$$= 102 \times 10^3 > 0 .$$

Therefore formula (183) applies and

$$t = 310 + \frac{1}{28.6\sqrt{4 \times 2.2408 \times 22 \times 10^{-6} - 0.09 \times 10^{-6}}}\left[\sin^{-1}\frac{2 \times 22 \times 10^{-6} \times 10.0 + 0.0003}{\sqrt{4 \times 22 \times 10^{-6}(22 \times 10^{-6} \times \overline{10.0}^2 + 0.0003 \times 10.0 + 2.2408)}} - \sin^{-1}\frac{0.0003}{\sqrt{4 \times 2.2408 \times 22 \times 10^{-6}}}\right]$$

$$= 310 + 4.8$$

$$= 315 \text{ seconds}$$

$$= 5.25 \text{ minutes.}$$